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ABSTRACT

In the first chapter Brownell critically examines the psychological bases of the three most common theories of arithmetic instruction: drill, incidental learning, and meaning. In chapter 2 the results of a nation-wide survey of actual teaching practices are reported. Chapter 3 presents a contrast between "informational arithmetic" and "computational arithmetic." In chapter 4 "social utility" is defined much more broadly than just "computationally useful," and implications for arithmetic instruction are discussed. Following this is a survey of opportunities for use of arithmetic in an activity program, with specific examples. Chapter 6 discusses practices in the teaching of fractions and decimals, followed by the report of a study on transfer by Overman. Reported in chapter 7 are the results of a survey on current practices in teacher-training courses in arithmetic. Two chapters are devoted to transfer of training in arithmetic and types of drill; using these as a basis, David Eugene Smith expounds on the past, present, and future of instruction in arithmetic. In chapter 11 arithmetic is considered from a mathematical viewpoint as contrasted to a pedagogical viewpoint. Next Gestalt psychology is discussed, with implications for mathematics teaching. The last chapter compares the efficiency of different methods for division. (LS)

THE NATIONAL COUNCIL
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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE TENTH YEARBOOK

THE TEACHING OF ARITHMETIC

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EDITOR'S PREFACE

This is the tenth of a series of Yearbooks which the National Council of Teachers of Mathematics began to publish in 1926. The titles of the preceding Yearbooks are as follows:

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The purpose of the Tenth Yearbook is to present some of the most important ideas and proposals concerning the teaching of arithmetic in the schools. Through this Yearbook the National Council of Teachers of Mathematics wishes to express its interest in the more elementary phases of mathematics and in the teachers who present it in the classroom.

I wish to express my personal appreciation as well as that of the National Council of Teachers of Mathematics to all of the contributors to this volume who have given so freely of their time and interest in helping to make this Yearbook worthwhile.

W. D. Reeve

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THE TEACHING OF ARITHMETIC

PSYCHOLOGICAL CONSIDERATIONS IN THE LEARNING AND THE TEACHING OF ARITHMETIC

By WILLIAM A. BROWNELL
Duke University

Arithmetic is singularly unfortunate in the language which has come to be used to describe the processes by which its subject matter is to be learned and to be taught. One continually encounters such terms as "the number facts," "skills," "consumers' arithmetic," "incidental learning," "automatic associations," "fixed habits," "bonds," "100% accuracy," "crutches," "meaningful experiences," "drills," etc., etc. These terms have a significance which is seldom sufficiently recognized, for in one way or another they imply certain theories regarding the psychology and pedagogy of arithmetic. More important, perhaps, they lead directly to the adoption of instructional practices of varying degrees of merit.

Analysis of ambiguous and misleading terms would repay the time required. Since space limitations forbid, however, another method is employed to get before the reader some of the crucial psychological aspects of learning and teaching arithmetic. This method consists in examining critically the psychological bases of the three commonest theories with respect to arithmetic instruction. To this examination the remainder of this chapter is devoted. The reader should be warned that a theory of arithmetic instruction is rarely, if ever, practiced in pure form. The theory held to predominantly by a teacher will determine the points of emphasis in her teaching practice, but the practice itself will reveal the influence of other and perhaps conflicting theories. The three theories, as they are here isolated for analysis, are not so isolated in practice. While it is probable that a particular teacher can be classified as adhering in general to one theory, she will not be found to be a one-hundred-per-cent practitioner of that theory. Rather, she tends, wittingly or unwittingly, to employ various features of two or even three of the theories. To examine the psychological foundation of the three theories, however, it is necessary to consider these theories separately—in purest state, as it were.

I. THE DRILL[1] THEORY OF ARITHMETIC

Exposition of the theory. The drill conception of arithmetic may be outlined as follows: Arithmetic consists of a vast host of unrelated facts and relatively independent skills. The pupil acquires the facts by repeating them over and over again until he is able to recall them immediately and correctly. He develops the skills by going through the processes in question until he can perform the required operations automatically and accurately. The teacher need give little time to instructing the pupil in the meaning of what he is learning: the ideas and skills involved are either so simple as to be obvious even to the beginner, or else they are so abstruse as to suggest the postponement of explanations until the child is older and is better able to grasp their meaning. The main points in the theory are: (1) arithmetic, for the purposes of learning and teaching, may be analyzed into a great many units or elements of knowledge and skill which are comparatively separate and unconnected; (2) the pupil is to master these almost innumerable elements whether he understands them or not; (3) the pupil is to learn these elements in the form in which he will subsequently use them; and (4) the pupil will attain these ends most economically and most completely through formal repetition.

Example of drill organization. The nature of the drill theory and of the four component aspects of the theory which have just been outlined may be made clearer by means of an illustration. The following section occurs in a textbook for the third grade. It was intended apparently to supply the child all he would need in order to learn the new topic, "Written Subtraction with Carrying." Following the section quoted from this book is a page of problems, none of which is analyzed, designed to provide drill on the new process.

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[1] The term "drill" is loosely used in discussion relating to arithmetic instruction. Sometimes it is employed in a sense which makes it cover all forms of instruction. Sometimes it refers only to practice and the maintenance exercises which follow initial instruction. And at still other times it signifies only maintenance activities. The reader should note carefully that in this chapter the term "drill" is used to characterize a theory of arithmetic instruction which makes repetition on the part of the pupil the essential feature of learning. In other chapters of the *Yearbook* the word "drill" will probably be used with a different signification.

1. Mary wants a doll that costs 42 cents. She has 28 cents. How much more does she need?

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42 Think, "8 and 4 are 12." Write 4. Carry 1 to 2.
28 Think, "3 and 1 are 4." Write 1.
14

Prove the answer by finding the sum of 14 and 28.
Think, "4 and 8 are 12." Carry 1 to 1.
Think, "2 and 2 are 4." 42 is the sum of 14 and 28.

When the number in any column is less than the one below it, think what added to the lower number will make a sum that ends in the number above, and carry 1.

It will be observed (1) that the process of carrying in subtraction has been isolated as one of the elements to be learned; (2) that meaning and understanding are neglected (with respect both to the problem as requiring subtraction and to the method of the process by which the subtraction is to be performed); (3) that the pupil is expected at once, without question, to take on a type of thinking characteristic of the expert adult who in meeting his practical need for subtraction performs the operation without thought of its underlying logic; and (4) that the pupil is to acquire the new skills by repetition—by working the examples and problems which follow without any assistance other than that provided by the mechanical model in the above quotation.

Popularity of the drill theory. Of the three conceptions of arithmetic instruction which will receive attention in this chapter the drill theory is by far the most popular. In the classroom its popularity is manifest in the common extreme reliance upon flash cards and other types of rapid drill exercises, in the widespread use of workbooks and other forms of unsupervised practice, and in the greater concern of the teacher with the pupil's speedy computation and correct answer than with the processes which lead to that computation and that answer.

But the popularity of the drill theory is by no means revealed only by the prevalence of certain practices in classroom instruction. On the contrary, its popularity is evident, as well, in the organization of arithmetic textbooks, in much of the research in arithmetic, in current practices in measuring achievement in arithmetic, and in treatises on the teaching of arithmetic.

The makers and publishers of arithmetic textbooks are prone to direct attention to the extraordinary care they have exercised with respect to drill provisions, as if the provision of adequate drill (that is, adequate according to certain criteria) were the crucial problem in the preparation of basic instructional matter. Research workers report laborious "evaluations" of textbooks in which, too frequently, differences in drill organization are divorced from differences in instructional theory which give significance to discovered variations in drill provisions. In commercial arithmetic tests and in treatises on educational measurement importance is given to set standards in terms of rate and of accuracy of computation, at the expense of growth in fundamental understanding, in orderly quantitative thinking, and the like. Last of all, few textbooks on methods of teaching arithmetic treat such matters as how children develop their number concepts or how they come to apprehend the rationale of the number system and the arithmetical processes it makes possible, preferring rather to deal with the selection of instructional items, the distribution of practice thereon, and the like.

Reason for popularity. The popularity of the drill theory is by no means to be understood as the result of a careful and intelligent judgment between the merits of the possible types of arithmetic instruction. More commonly the drill theory is adopted and practiced, one may conjecture, without any clear comprehension of its assumptions and implications. Wide acceptance of the theory seems to be due to two misleading approaches to a definition of arithmetic ability: (*a*) analysis of adults' uses of arithmetic and (*b*) the "bond" theory of learning.

(*a*) The theory draws support (or seems at first glance to draw support) from the adult's everyday use of arithmetic. Thus, in making a purchase of a loaf of bread for six cents and of five pounds of sugar for twenty-five cents, an adult seldom, if ever, hesitates in finding the total. Much less does he inquire into the reason why 6 and 25 are 31, or into the methods of the thinking by which he secures the sum. The arithmetic teacher, performing daily many such computations, is struck by the automatic, the instantaneous quality of her reactions. From this observation of her own conduct it is but a step to the conclusion that since she *uses* number so, so the child should *learn* it. She has quite forgotten her own trials in learning arithmetic. She probably does not even realize that she never did learn "25 and 6 are 31" as a number fact.

(*b*) Nor does the teacher who looks beyond her own present number experience find much assistance in the typical psychological treatments of arithmetic instruction. As a matter of fact, the drill theory of arithmetic has become popular largely because of the popularity of that system of psychology which has been most influential in education in the last two decades. According to this school of psychology *all* learning consists in the establishment of connections or bonds between specific stimuli and specific responses. This view of learning in general seems to be of particular value in describing the learning process in the case of number. Thus, one connects the response "6" with the stimuli "4 and 2," the response "9" with stimuli "3 × 3," and so on. Each arithmetic fact represents such a bond, and all skills likewise are reducible in last analysis to similar bonds. It seems to follow, therefore, that the way to teach arithmetic is to teach immediately and directly the bonds which are to be established. Drill then becomes the instructional method best adapted to this end, and repetition becomes the essential mode of learning.

In view of the prominent place given to repetition by the drill theory, the rest of this section is given over to a consideration of the rôle of repetition, first, in learning in general and, second, in arithmetical learning.

Repetition in learning in general. As contained in certain formulations and interpretations of the Law of Exercise, repetition was for many years regarded not only as a factor but as the important factor in promoting learning. To repeat a reaction was to learn it. To acquire a form of behavior, one had to repeat it. To the extent that one repeated, one learned. More recently, under pressure alike from critical theoretical discussion and from research findings, the Law of Exercise has been restated by many psychologists and has been entirely discarded by others. Now other factors than repetition have come to be viewed as the vital determinants of learning.

An example from ordinary experience will reveal the place and function of repetition in learning. Suppose Mr. B. undertakes to improve his "drive" in golf. He is observed to set up his ball, to grasp his club, to take a position with respect to ball and line of flight, and to swing. Suppose further that the result of the first swing is not very successful. Does Mr. B. "repeat" his first performance on his second try? Does he set up the ball exactly the same height, does he stand the same distance from the ball, does he place his hands precisely as he did the first time, and so on? Hardly: certainly not if he can help it. Instead of attempting to "repeat," he does his best to avoid repetition. He varies, changes, modifies—all with a view to securing a new combination of reactions which will produce more satisfactory results. Suppose, however, he does repeat his first unsuccessful performance. He stands, holds the club, and swings exactly as before. The result is the same poor shot, but made a little more proficiently. If he continues to repeat, he becomes steadily more expert in poor golf. If he would improve his game, he must cease repetition and adopt variation. The point of the illustration is that repetition can at most increase only the speed and the accuracy of a reaction, good or poor; it cannot furnish a new way, a better way, of doing anything.

Criticism of the drill theory in arithmetic. Three major objections may be raised to drill as the sole, or even the principal, method of arithmetic instruction. The first objection is that the drill theory sets for the child a learning task the magnitude of which predetermines him to failure. The second objection is that drill does not generally produce in children the kinds of reaction it is supposed to produce. The third objection is that, even if under conditions of drill the proposed kinds of reaction were implanted, these reactions would constitute an inadequate basis for later arithmetical learning.

(*a*) *Magnitude of the task.* No one can define the limits of the learning task in arithmetic when that task is described in terms of the drill theory. Each item of knowledge or skill must, according to the theory, be isolated and specifically taught. The objectives of arithmetic set by this theory become utterly unattainable. Consider, as a case in point, the situation with regard to the number of addition combinations which must be taught. Years ago there were assumed to be 45 simple addition facts, the mastery of which, accompanied by an understanding of addition by endings, was regarded as sufficient preparation for all phases of this process. Later, research established the fact that children who knew $4 + 5 = 9$, for example, might not know $5 + 4 = 9$ equally well. The reverse combinations had, therefore, to be taught, and 36 new combinations were added to give a total of 81 facts. Investigation then demonstrated peculiar difficulties in the zero-combinations, and the addition combinations to be taught grew from 81 to 100 to accommodate the nineteen zero-facts. More recently it has been shown that the knowledge of $5 + 4 = 9$ does not guarantee the knowledge of $15 + 4 = 19$, $25 + 4 = 29$, or $45 + 4 = 49$, etc. To the 100 simple addition combinations there now have been added, at Osburn's[2] suggestion, the 225 combinations required by higher-decade addition with sums to 39 and the 87 combinations required for carrying in multiplication. The original 45 addition combinations have now become 412.

Similar analyses of other phases of arithmetic have resulted in item-totals equally staggering. Thus, Knight demonstrates the presence of 55 "unit skills" in the one process of division of common fractions, and Brueckner finds 53 "types" of examples (exclusive of "freak types") in the subtraction of common fractions alone. In problem-solving Judd reports, on the basis of an examination of only three sets of textbooks, approximately 1,900 different ways of expressing the fundamental operations in one-step verbal problems, and Monroe and Clark are able to differentiate 333 kinds of verbal problems (52 kinds of "operative problems" and 281 kinds of "activity problems") with which children must eventually be able to deal. These figures, large as they are, can be regarded only as typical of what would be found if the many other aspects of arithmetic were dissected as have been the relatively few reviewed above.

The statement that the drill theory in its extreme form sets an impossible learning task for the child would seem to be justified by the results of the analyses mentioned above. If the child must learn separately and independently, through repetition, in the form in which he is later to use them, all these items of knowledge and skill (and tens of thousands as yet untabulated), what are his chances of success in arithmetic? Suppose the reader were faced by the necessity of mastering an equivalent number of unrelated meaningless items—not only to master them, but to remember them, to retain order among them, and to use them intelligently when and as they should be used. Assume such a condition—what chance of success would the reader have in this learning? Would he even be willing knowingly to undertake the task?

Now manifestly, no one actually carries the drill theory to the extremes suggested in the foregoing paragraphs. The teacher, with or without aid from textbook and manual, does supply something of a sensible basis for learning, and thereby breaks with the logical requirements of the drill theory.[3] Even the theoretical writer on arithmetic, who advocates repetition as the basis of arithmetic learning, is in the end forced to be inconsistent with his own view. Soon or late he must call to his assistance the factor in learning which his theory in effect denies. At the last he too recognizes the impossibility of training children in all the separate skills required in the total which he knows as arithmetical ability. In his dilemma the theorist finally finds himself invoking transfer of training to guarantee that children will be able to deal adequately with skills they have had no time to learn as such.

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[2] The reader will understand, of course, that the citation of these analyses does not classify their makers as exponents of the drill theory.

(*b*) *Reactions produced by drill.* When the teacher provides drill in arithmetical skills, she does so on the assumption that pupils will exactly practice certain prescribed reactions. Thus, for example, when she administers flash-card drill on such number combinations as $4 + 3 = 7$, $8 - 6 = 2$, etc., she expects all pupils to think silently or to say aloud, "4 and 3 are 7," "8 less 6 are 2," and so on. It is her belief that by such repetition the children will come eventually to respond only and always "7," "2," etc., on presentation of the corresponding combination items. To express the idea differently, the administration of drill by the teacher presupposes repetition by the pupil. That this presupposition, this assumption, is not warranted by the facts is well shown by the results secured in a recent investigation.[4]

About a week after the beginning of the school year fifty-seven third grade children were given a written test in the 100 simple addition combinations. They had been taught these combinations in Grades 1 and 2 by methods which agree closely with the drill theory of instruction.[5] On the basis of their showing on this group test thirty-two children were selected for individual study. Those chosen were ten who had the highest scores, thirteen who had average scores, and nine who had very poor scores. An interview was held privately with each child. In these interviews an attempt was made to discover how each child secured his sums, that is, how he thought of the numbers, what processes he employed. For the interview sixteen combinations were used. These sixteen consisted of the ten of greatest difficulty and the six of average difficulty on the group test. There were, therefore, a total of 512 responses in the interviews—the responses of each of the thirty-two children to sixteen combinations.

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[3] Here is, by the way, a good example of the distortion against which a warning was given on page 1. Such a discussion as this must exaggerate conditions in order to make clear important differences.

[4] Chazal, Charlotte B., "The Effects of Premature Drill in Third-Grade Arithmetic." Unpublished A.M. thesis, Department of Education, Duke University, 1935. An abstract of the thesis under the same title is to appear shortly in the *Journal of Educational Research*.

[5] Specifically, the facts were taught as facts from the outset. Experience with concrete numbers was kept at a minimum. The facts were not developed for the child or discovered by him. They were given to him, that is to say, he was told that $3 + 4$ are 7, and was then drilled on this fact along with similarly presented facts. The usual variety of drill activities was provided: there was repetition by the class as a whole, and repetition by individual children. There were oral drills and silent drills and written drills. Packs of cards were handled, the one side

The interview revealed that 116 combinations (22.7% of the 512) were *counted*; that 72 (14.1%) were *solved indirectly* (e.g., "6 and 4 are 10 because 5 and 5 are 10"); that 122 (23.8%) were incorrectly *guessed*; and that only 202 (39.5%) were *known* as memorized associations. These facts can be interpreted only as meaning that the instructional procedure of drill had missed its mark in Grades 1 and 2. It had utterly failed to produce the 100% of "automatic responses" for which it was designed, and it had failed by the wide margin of 60%. After two years of drill these pupils counted and solved nearly as many combinations (36.7%) as they knew directly as combinations (39.5%). The evidence is that, expected to repeat the formulas, these pupils had not repeated at all. Unknown to the teacher who assumed they were repeating, they had trained themselves in other ways of thinking of combinations. Drill by the teacher had not resulted in repetition by the pupils.

The investigation did not stop here. In the month following the first group test and the first interview five minutes of the arithmetic period were each day set aside for drill on the addition combinations. The drill, which called for oral, silent, and written practice, was so organized that, on the average, each combination was presented at least twice a day. Then came the second administration of the group test on the 100 addition combinations and the second interview with the same thirty-two children on the same sixteen combinations. At this time 48.8% of the combinations, as

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containing the combinations, the other side, to be consulted only if the sum was unknown, containing the answer. There were games to motivate drill, etc., etc. In all the activities, be it noted, the essential requirement on the part of the child was that of *repetition*, however much variation there may have been in the form of the response or in the number presentations which evoked that response.

compared with 39.5% on the first interview, were ***known***, that is to say, were responded to as they are supposed to be responded to under drill conditions. On the other hand, ***counting*** and ***indirect solution*** still accounted for 37% of the answers (as compared with 36.7% on the first interview). The evidence from the second phase of the investigation agrees closely with that from the first phase: drill was provided by the teacher, but repetition was *not* provided by the children. Instead, the children continued throughout the month of drill to employ substantially the same procedures in thinking of the combinations as they had developed in Grades 1 and 2 and as they brought with them into Grade 3. If at the time of the first interview they counted their combinations they persisted in ***counting*** a month later, in spite of the daily drill which was designed to require repetition. If at the time of the first interview, they ***solved*** the combinations, they ***solved*** them a month later. Drill failed signally to produce in these children the desired types of mastery.

The study just described deals, it is true, with drill in connection with but a single phase of primary grade arithmetic. There is, however, no reason to doubt that the weakness here ascribed to repetition as a method of learning and to drill as a method of teaching holds with any less validity for more advanced phases of arithmetic.

(*c*) *Preparation through drill inadequate.* The third criticism of the drill theory arises from a consideration of the nature of arithmetic itself. Arithmetic is best viewed as a system of quantitative thinking. To describe arithmetic in this way is to set up a criterion by which to judge the adequacy of any system of arithmetic instruction. If tried by this criterion, the drill theory is found wanting: instruction through drill does not prepare children for quantitative thinking.

If one is to be successful in quantitative thinking one needs a fund of meanings, not a myriad of "automatic responses." If one is to adjust economically and satisfactorily to quantitative situations, one must be equipped to understand these situations and to react to them rationally. Specific responses to equally specific stimuli will not serve one's ends. It may be granted (in spite of the evidence to the contrary in the preceding section of this chapter) that drill may furnish to the child a number of fixed modes of response. Even so, its claim to preëminence as a method of instruction in arithmetic would have to be denied. Drill does not develop meanings. Repetition does not lead to understandings.

This limitation of drill may be illustrated by citing again the learning of the number combinations. Suppose that a pupil, through repeating the formula, has memorized "12" as the answer to "How many are 7 and 5?" Suppose, further, that in the absence of other types of experience than repetition, the pupil is asked, "What does it mean to say that 7 and 5 are 12?" His reply must be, "I don't know—just that 12 is the answer to 7 and 5." The meaning of "7 and 5 are 12" is for him restricted to merely making the appropriate noises and to reading and writing the symbols which stand for the combination.

The psychological fact is that meanings are dependent upon reactions. It is not too much to say that the meanings *are* the reactions. Meanings are therefore rich and full and useful to the degree to which the corresponding reactions have been numerous and varied. To repeat "7 and 5 are 12" is to practice but a *single* reaction. No matter how long continued or how frequent the repetition the effect is only to increase the efficiency (the rate and the accuracy) with which that reaction is made. Increased efficiency in making a small number of reactions is no substitute for rich meanings. If there are to be meanings and understandings (and there must be these if children are to be capable of quantitative thinking), there is but one way to engender them. That way is to lead children to react variously and often to the item of knowledge or skill which is to be acquired.

As stated above, drill and repetition are ill-adapted to the building of meanings. In them the pupil finds no suggestion of a variety of reactions, no help and no encouragement to discard primitive and clumsy ways of thinking (immature reactions) for steadily more refined and efficient thought processes (mature reactions). On occasion, drill may, it is true, be given credit for accomplishing just this end. Such seemed to be the case, for example, in the investigation referred to in the foregoing section.[6] The month of drill on the combinations which followed the first group test brought a reduction of approximately 35% in the amount of time required to write the sums for the 100 addition combinations and a reduction of 75% in the number of errors made. The interview data, however, contained the real explanation. Drill had not developed improved procedures: it had merely afforded opportunity for practice and increased efficiency with undesirable procedures. Under other circumstances, too, drill may be given undeserved credit for a different reason. Children may despair of ever mastering by repetition all the subject matter of arithmetic. In the end, quite without the knowledge of the teacher who continues to prescribe drill, they may desert the repetition of verbalisms in the effort to put order and sense into what they are learning. They may discover or adopt from others modes of response which actually do increase the number of ways in which they may react to the items to be learned. Under such conditions if arithmetic becomes meaningful, it is absurd to assign the credit to drill. It is much nearer the truth to say that if under these conditions arithmetic becomes meaningful, it becomes so in spite of drill.

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[6] Chazal, *op. cit.*

II. The Incidental Learning Theory

Nature of the theory. At least partly as a reaction against the forcing and driving classroom tactics of teachers who are zealous advocates of the drill theory, a second theory, or group of theories, has become increasingly popular in the last fifteen years or more. According to these theories, which differ chiefly in detail, children will learn as much arithmetic as they need, and will learn it better, if they are not systematically taught arithmetic. The assumption is that children will themselves, through "natural" behavior in situations which are only in part arithmetical, develop adequate number concepts, achieve respectable skill in the fundamental operations, discover vital uses for the arithmetic they learn, and attain real proficiency in adjusting to quantitative situations. The learning is through incidental experience. The theory is accordingly here designated the "incidental learning theory."

Some who hold to the theory of incidental learning would postpone systematic arithmetic instruction until the third or fourth grade and would concern themselves not at all with the kind of number usages, if any, which children encounter before that time. Others differ from this position only in that they would postpone arithmetic instruction much longer (one investigator reports the omission of "formal" arithmetic through the seventh grade without "harmful" consequences in the eighth grade). And still others, distrusting somewhat the wholly hit-or-miss number contacts which would result from the first two variations of the theory, would select arithmetical activities for children, but would arrange them in such a way that the arithmetic is only a minor part of the total situations in which children might find themselves. Exponents of this last plan are those who prepare "integrated units" of activity, in which the different subject-matter fields, including arithmetic, of course, lose their identity.

Criticism of the theory. Critical comment here must be confined to two points which are common to the different variants of the theory of incidental learning. The first of these points involves the place of interest in learning. The second involves the assumption that children will themselves isolate the arithmetical aspects in general situations and that through purely incidental number experiences will attain whatever arithmetic ends and outcomes are set for them.

(*a*) *Place of interest in learning arithmetic.* The theory of education through the process of incidental learning probably had its origin in the discussion attendant upon the emergence of the "child-centered school." Some years ago the more radical leaders in this reform movement did not hesitate to insist that all education must be derived from, and must be organized and directed with respect to, children's self-determined interests and needs. This extreme position, which is held less widely now than formerly, rested upon a faulty psychological basis. Interests and needs were supposed to arise spontaneously, as the product of some kind of inner compulsion which made their appearance more or less automatic at a predictable time. Viewed thus, interests are sacred; they are not to be altered, for alteration amounts to profanation. To disregard these natively fixed indices of growth, or, worse yet, to go against them, is to "violate the child's nature."

It is now rather generally recognized as a fact that children's interests are socially determined. The reason why children of a given age manifest certain interests is found, not in heredity, but in growth under conditions which are relatively alike for all children. Stated differently, children's interests and needs are but the products of experience. As such they reflect the useful, the valueless, and the harmful in that experience. Viewed thus, interests and needs lose much of their sacred quality. Modification and direction are no longer to be denied. On the contrary, they must be exercised if growth and development are to be sound and healthful.

The foregoing paragraphs may seem to dispose of a still debatable issue in a manner which is overly abrupt and summary. If so, this has been done only to clear the way for a more fundamental criticism of the place accorded interest by the incidental theory of arithmetic instruction.

Probably no one today doubts the virtues of interest as a factor which facilitates learning. On the contrary, all agree that children learn best when they want to learn. So far as effects on learning are concerned, the source of that interest, whether it be heredity or society, is of little consequence. Even the ardent practitioner of the drill theory seeks to have his pupils interested. (Witness his attempt to discover races, games, and other means for motivation.) Be that as it may, it is to the credit of those who have expounded the theory of incidental learning that the importance of interest in learning is now more generally recognized. These theorists have consistently and vigorously preached that pupils should want to learn what they are given to learn. And they have made real headway with their gospel.

Two objections may be entered, however, to the part assigned to interest by the incidental theory of teaching arithmetic. The first objection is that from the standpoint of learning arithmetic the pupil's interest is apt to be in the wrong place. When number is observed only as it functions in situations which are predominantly non-arithmetical, there is little likelihood of the child's being interested in number. That is to say, he is interested in the learning situation, but, for the purpose of arithmetic learning, he is interested in the unimportant part of it. He is primarily concerned with the successful completion of his unit of activity or of his project. If he is interested in the arithmetic at all, the interest is secondary and derived. He gives his attention to the arithmetic only because of the extrinsic motivation furnished by the situation. Under these conditions his interest in arithmetic as such is apt to be as superficial as is his interest when it is stimulated in drill through games and similar devices. The pupil learns as the adult learns—where his interest lies. His interest lies in the larger activity as such, and consequently he acquires, first of all, types of behavior which relate most closely to that activity. What interest is left over may or may not attach itself to arithmetic.

The second objection to the place of drill in the incidental theory is the implication that interest aroused in any way other than through large units of purposeful activity is somehow unworthy. There seems to be the notion that the desire to learn arithmetic for its own sake, that is, through intrinsic motivation, is somehow unnatural and something to be discouraged as being in some way undesirable. Admittedly, such an interest is hardly apt to appear under conditions of incidental learning, but this fact cannot be argued to mean that the interest is, on that account, unfortunate or any the less effective as a factor in learning. The contrary is the truth. It is a wholesome situation, if not a common one, for children to want to learn arithmetic because they like it. Under the stimulation of such a motive, moreover, it is highly probable that the learning will be economical and thorough. And, last of all, this happy state of affairs can be brought about through a direct instructional attack upon arithmetic as a subject for learning, fully as well, if not better, than by the indirect and uncertain experiences prescribed by the incidental theory.

(*b*) *The product of incidental experience.* When an individual reacts to some general situation in which there is a quantitative element, the form of his response to that quantitative element will depend upon many things. Chiefly, perhaps, it will depend (1) upon his preparation for reacting to such elements and (2) upon his disposition with regard to the element at that time. In the case of the educated adult, who has *learned* his arithmetic, the quantitative element may be readily isolated, correctly apprehended, and promptly dealt with. The adult is only momentarily distracted from his major objective, to which he returns at once upon solution of the quantitative problem. He has learned no new mathematical ideas or operations; he has merely applied previously acquired ability in a new connection.

The experience of the child as he engages in an assigned "integrated unit" or in his self-chosen and self-controlled activities may be quite different from that of the adult. If the general situation which he faces requires no *new* arithmetical skill or knowledge, his experience may be quite like that of the adult. Like the adult, too, he will apply to the situation only what he has already learned; he will learn nothing new in arithmetic from his experience. If, on the other hand, the situation involves some unknown arithmetical skill or knowledge his experience will be quite unlike that of the adult. The child's concern is with the attainment of some end beyond the intermediate arithmetical task. Too long delay in mastering the novel arithmetical skill is to him unthinkable—it may be ruinous to his plans. At best he gives to the arithmetical aspect of the situation only enough attention to remove it as an obstacle to the realization of his purpose. Furthermore, he deals with that quantitative element with whatever habitual response he may possess or with a response altered as little as may be.

It is at this point that the teacher who relies upon the incidental theory comes, curiously enough, very close to one of the fallacies of the drill theory which, as part of that theory, she would oppose. That is to say, she is very apt to make the error of assuming that mere contact with number teaches all that the child needs to know about number. She may easily overlook the fact that the quantitative aspects of general situations represent only possibilities for learning but not guarantees of learning. The child in the first three of four grades has few "natural" needs for arithmetic which he cannot meet by counting. Left to himself he will solve his problem by counting. Consider, for example, the following situation: the child is constructing a boat. He must fasten an irregularly shaped board to the side of the boat. He needs four nails for one end and three nails for the other end. To the adult the situation calls for the use of the fact "4 and 3 are 7." To the child the situation merely calls for "4, 5, 6, 7," or even "1, 2, 3, 4, 5, 6, 7." He does not see the situation as involving "4 + 3" at all. Counting is to him a wholly satisfactory method of dealing with the number need. So he counts, regardless of what his teacher thinks he should do.

Unquestionably, opportunities for arithmetical learning abound both in the practical affairs of children and in "integrated units," just as the supporters of the theory of incidental learning insist. If, however, these opportunities are to be utilized for the development of sound skill and knowledge, the new arithmetical elements must be abstracted from general situations and must become the objects of direct teaching. In this way provision can be made for supplementary experiences which, by introducing variety and number of reactions, guarantee the meaningful concepts and the intelligent skills requisite to real arithmetical ability.

There is a second way in which instruction based upon the incidental theory is likely to disregard the peculiar nature of arithmetic learning. To a far greater extent than is true in the case of any of the other school subjects (at least as these subjects are now known) arithmetic must be taught with due regard to its logic, its internal order and organization. By contrast, there seems to be no compelling reason why in history, for example, Topic A must be taught before Topic B. Justifiable violations even of chronological sequence are frequent, and events and personages may be omitted or treated in detail without serious consequences which are discernible. Similar illustrations could be drawn from geography and others of the content subjects. But in the case of arithmetic the number of changes which can safely be made are comparatively few. Integers must be taught before fractions, and the combinations must be taught before complicated operations which make use of them. This internal organization must be observed not merely with respect to the relatively fixed order of topics but also with respect to the sequence within the topics. Thus, within the process of addition the various skills and sub-skills must be arranged so that mastery of a late step is made possible by mastery of preliminary steps. This internal coherence and organization are determined on the one hand by the nature of the subject matter and on the other hand by the psychology of the learner.

It is just this organization which is difficult to maintain when the learning of arithmetic is left to the incidental experiences of children. This statement holds even when these experiences are prescribed in "integrated units."[7] When, however, there is no attempt thus to select number experience, the likelihood that children will learn in what is psychologically and logically the most economical fashion is extremely small. As a consequence, arithmetic can hardly be learned as it should be learned; relationships, dependencies, mathematical principles may easily escape the notice of the teacher, and so of the learner.

Impracticability of the theory. All that has been said in the above section helps to explain why arithmetic instruction based upon the incidental theory of learning is impracticable. Incidental learning, whether through "units" or through unrestricted experiences, is slow and time-consuming. Interest of teacher and pupil alike in the non-arithmetical aspects of general situations tends to reduce the occasions on which the arithmetical aspects are called to the learner's attention. Such arithmetic ability as may be developed

[7] This statement does not imply that number needs *appear* in the life of children with due respect to the order prescribed by mathematical logic. Such is not the case. The statement applies to *learning*. Even though a desirable mathematical sequence is disregarded in the time when number uses appear, that mathematical sequence must be, in large measure, honored in the instruction given in connection with these uses.

in these circumstances is apt to be fragmentary, superficial, and mechanical.

However successful an occasional teacher may be in teaching arithmetic through incidental experience, general attainment of this success is not possible. The discriminating selection and orderly arrangement of vital and helpful learning situations involving number is no simple task. On the contrary, it calls for unusual insight into the mathematical and psychological nature of arithmetic on the one hand and into the psychology of childhood and of the learning process on the other hand. In a word, it calls for a degree and kind of insight which is, without aspersion of teachers as individuals, quite outside the equipment of the average teacher. Until teachers are differently selected and differently trained, it is fruitless to expect them adequately to teach children arithmetic through incidental experience. In the meantime it is far safer to base arithmetic instruction upon a judicious use of textbooks in the preparation of which the kind of insight just described has been exercised.

Values of the theory. Like the drill theory of arithmetic instruction, the theory of incidental learning is not without certain advantages and merits which should be recognized and utilized to the full. These special values are at least three in number.

In the first place, the experiences of children in situations which are only incidentally arithmetical (whether in "units" or otherwise) may serve as powerful motives for the learning of new arithmetical ideas and processes by revealing the need for such abilities. In the second place, the opportunities for practice afforded in the general situations which children face both in and out of school may have the effect of increasing the meaning of number ideas and skills which have already been acquired and of maintaining them at a high level of efficiency. In the third place, perhaps the most important contribution of the theory of incidental learning has been to oppose the common practice of teaching arithmetic narrowly as an isolated subject. The incidentalists have rightly argued that arithmetic so taught cannot perform its full or, indeed, its most valuable function, for number is a real socializing agency.[8] Arithmetic provides an exact method of interpreting practical quantitative problems which otherwise are confused and unintelligible. On this account instruction in arithmetic must lead children to discover and to use the number they learn, not merely in order that they may know it better but in order that they may attain ends which are largely outside of arithmetic. It is at least an implication of the theory of incidental learning that children do not know arithmetic as they should until they are better able to prosecute their own designs and to understand the quantitative aspects of the society in which they live.[9]

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[8] That this view of arithmetic is by no means original with the advocates of incidental learning is clear from a perusal of the writings of Charles H. Judd who, as long ago as 1903 in his *Genetic Psychology for Teachers* (Chapter IX, especially) was expounding this wider interpretation of arithmetic.

III. THE "MEANING" THEORY OF ARITHMETIC INSTRUCTION

The third theory of arithmetic is not readily named. It can hardly be termed the "eclectic" theory because, while it does contain features of other theories, it also contains features which are peculiarly its own. Furthermore, this third theory is too coherent and unified to justify the implication of looseness and forced relationships which is associated with the word "eclectic." With this acknowledgment that no name would be without objectionable connotations, the third theory is here designated as the "meaning" theory. This name is selected for the reason that, more than any other, this theory makes meaning, the fact that children shall see sense in what they learn, the central issue in the arithmetic instruction.

Relation to other theories. Within the "meaning" theory the virtues of drill are frankly recognized. There is no hesitation to recommend drill when those virtues are the ones needed in instruction. Thus, drill is recommended when ideas and processes, already understood, are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse. But within the "meaning" theory there is absolutely no place for the view of arithmetic as a heterogeneous mass of unrelated elements to be trained through repetition. The "meaning" theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes. According to this theory, the test of learning is not mere mechanical facility in "figuring." The true test is an intelligent grasp upon number relations and the ability to deal with arithmetical situations with proper comprehension of their mathematical as well as their practical significance.

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[9] In this chapter not much attention is given to the social phases of arithmetic. The brevity of this treatment does not imply any doubt as to their importance, which cannot be questioned. Fuller discussions will be found in the third and fourth articles, the presence of which in this *Yearbook* constitutes the reason for omission in this article.

There is room, also, in the "meaning" theory for certain features of the theory of incidental learning. The "meaning" theory allows full recognition of the value of children's experiences as means of enriching number ideas, of motivating the learning of new arithmetical abilities, and especially of extending the application of number knowledge and skill beyond the confines of the textbook. But the efficacy of incidental learning for developing all the types of ability which should be developed in arithmetic is held to be highly doubtful by advocates of the "meaning" theory.[10]

Encouragement of understanding. It has been said that the "meaning" theory is designed especially to encourage the understanding of arithmetic. It does this in at least three ways.

(*a*) *Complexity of arithmetical learning.* First of all, it takes full account of the complexity of arithmetical learning. The significance of this statement may best be appreciated by contrasting various ways in which certain arithmetical ideas and skills may be taught. In the following paragraphs the "meaning" approach to number concepts and to the number combinations is presented in some detail.

Arithmetic, when viewed as a system of quantitative thinking, is probably the most complicated subject children face in the elementary school. Number is hard to understand because it is abstract. No special "arithmetic instinct" fits the child directly to learn arithmetic. Neither does nature provide the child with tangible evidence of number which he can apprehend immediately and thus come easily to know through sense perception. There is no concrete quality of "five-ness" in five dogs which may be seen, heard, and handled. Color, barking, weight, and shape may be grasped through the senses, but the "five-ness" is not thus open to immediate observation. Neither is there any "five-ness" in ⁙, or in "five," or in "5." In each case the "five-ness" is the creation of the observer: it is a concept or an idea which the observer imposes upon the objective data. Furthermore, it should be clear that the observer cannot impose the number idea "five" upon objects unless he has that idea—unless he has acquired the thought pattern which stands for "five." Such considerations as these with regard to the nature of arithmetic reveal the fact that from the very start, in the earliest as well as the later grades, number is complex.

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[10] These statements should make it clear that the "meaning" theory is no compromise. It does not represent an attempt to harmonize differences in the drill and the incidental theories. The "meaning" theory is a separate theory, which stands or falls on its own merits or weaknesses.

One way of putting "five-ness," "seven-ness," "ten-ness," etc., into objective representations of number is to enumerate. The ability to count objects the school does develop, but it does little more than this by way of providing children with other, and more advanced, ways of thinking of concrete numbers. Too commonly instruction in counting is immediately followed by drill on the addition and subtraction combinations.

This approach to primary number almost totally neglects the element of meaning and the complexity of the first stages in arithmetical learning. It even disregards the evidence provided by children themselves that they do not understand what they are learning and that they are in trouble. When children know a combination one day and do not know it the next, there is something wrong in the learning. So is there something wrong when, told that their sums and remainders are wrong, children complacently make other errors. Likewise, there is evidence of difficulty and of faulty learning when children's written responses reveal such situations as this $\begin{array}{r}3\\+2\\\hline\end{array}$ ::· , and when their oral responses are delayed while resort is made to counting and to other undesirable procedures. According to the "meaning" theory these evidences of difficulty must not go unheeded.

The truth is that training in counting alone is insufficient to develop number ideas. Assume that the child has correctly counted five given objects. What has he found out? Perhaps very little indeed. It is true that he has employed the sequence of number names accurately in a one-to-one correspondence with the objects, but the "five" he announces at the end may mean merely that he has run out of objects and that consequently he has no more verbal responses to make. There is no quantitative significance in such counting; the child might as well be saying, "a, b, c, d, e," as "1, 2, 3, 4, 5." Or the child may mean by the "five," not the group but the last object, the fifth one. Or again, if he means by "five" to indicate a total, that total is constituted only of discrete ones; "five" is thus one, and one more, and one more, and one more, and one more. The "five" in such a case stands for no *group*, for no *unit*, for no single pattern in his thinking. It is but a conglomeration, a loose organization, of ones. Before this child is ready to deal understandingly with situations involving grouping, he must learn to see numbers as groups.

Accordingly, the "meaning" theory interposes a definite period of instruction between counting and the number combinations. The purpose of this period of instruction is to provide for the child activities and experiences which will carry him by easy stages from enumeration to meaningful ideas of numbers as groups. The child begins with concrete number—with objects which he can see and handle. He makes groups of objects, compares groups of objects, estimates the total in given groups of objects, learns to recognize at a glance the number of objects in small groups and in larger groups when the latter are in regular patterns.[11] Eventually he comes to think of concrete numbers in terms which are essentially abstract. At the conclusion of this period of learning, "5" is as much a unit in his thought processes as is "1." The "5" does not need to be broken down into five 1's. It is a meaningful concept and is available for use as such in new relationships. Equipped with this and other like number concepts, the child is ready for the number combinations.

If the number combinations were "number *facts*" as they are frequently said to be, children would encounter little difficulty in learning them. They can easily learn "two dogs and three dogs are five dogs," for this is a fact. But "2 and 3 are 5" is not a fact; it is a generalization. If it were a fact, children could, as drill advocates desire them to do, memorize it as they would a fact in history. Since, however, it is a generalization, the learning is much more arduous and much more time-consuming. One learns the number combinations as he learns other generalizations, not all at once by some stroke of will or mind, but slowly, by abstracting likenesses and differences in many situations, by reacting to the number aspects of situations in steadily more mature ways.

As stated in the criticism of the incidental theory, the presence of three objects and of five other objects in the same situation does not automatically suggest to the child "$3 + 5 = 8$." If the child is to think of the "3" and the "5" in the form of an abstract combination, he must be taught to see it so. It does little good to tell him that 3 and 5 are 8. He will have to be told the same thing again the next time the situation, or one like it, occurs, or else he will memorize the statement. Memorization at this time should [illegible] all means be prevented. Instead of telling the child "$3 + 5 =$ [illegible] and of urging him to memorize it, the teacher should lead the child to *discover* it. Furthermore, one discovery is not enough. He must discover it many times and in connection with many situations. At the beginning he will need to make the discovery with concrete materials. Eventually he will rediscover the same relation in abstract numbers. Too, his method of discovering the fact will change. He may have to count at first. This type of reaction should not be forbidden if it is necessary to the child, for it may be his only means of relating the numbers. As fast as may be, however, he should be helped to eliminate counting in favor of some more mature method of dealing with the numbers. Thus, he may see that 3 and 5 are 8 because the objects may be repatterned as $4 + 4$, or as $2 + 6$, or what not. Finally, he should come (and under skillful teaching he will come) to the point where he reaches the generalization, $3 + 5 = 8$. Now is the time for him to memorize the fact, if, indeed, he needs to memorize it. It is far more likely that his numerous and varying experiences with these number relations will have been enough to fix the fact for him without memorization. Drill will, however, be of service in increasing facility of recall and in assuring permanence to the learned fact.

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[11] For an example of the kind of primary number instruction which is here described in general terms, see: Deans, Edwina, "The Effect of the Meaning Method of Instruction on the Teaching of Second-Grade Number." Unpublished A.M. thesis, Department of Education, Duke University, 1934.

(*b*) *Pace of instruction.* In the second place, understanding of arithmetic is encouraged, in the "meaning" theory, through adapting the pace of instruction to the difficulty of the learning. At first when the new ideas and processes are unfamiliar the pace is kept slow. Time is allowed for meanings to develop before children are expected to employ the given item of knowledge or skill as a highly habituated reaction. To some extent this feature (adaptation of instructional pace to learning difficulty) has been illustrated in the case of the simple number combinations. It may be further illustrated by considering instruction in the case of the addition combinations with sums above 10.

The common practice in teaching $7 + 5 = 12$, for example, is to provide the child with a single picture of two groups of objects, seven and five in number. He is then asked how many objects there are in all. Not being given any way of securing the total, he probably counts the parts. He is then informed that his answer "twelve" is correct, and the fact is written for him as $\begin{array}{r}7\\+5\\\hline 12\end{array}$ or as $7 + 5 = 12$. He then proceeds to "learn" it, that is, to repeat it, until it has taken its place among the dozens of similarly memorized items.

The "meaning" theory outlines quite a different kind and pace of instruction. The child's difficulty in attempting finally to habituate the combination is recognized as due to causes which relate to the initial stages of learning. Accordingly, the rate of instruction at first is kept slow. Activities and experiences containing the new fact $7 + 5 = 12$ are multiplied. Furthermore, the child is not left at the primitive level of counting as his only means of understanding the relationship. Instead, he is soon shown how to complete the first number (7) to 10 by taking from the second number (5), and thus to translate the new fact into a familiar one ($10 + 2 = 12$). He first discovers the identity of $7 + 5$ and $10 + 2$ by using concrete objects. He rediscovers them with other concrete objects and later with pictures, with semi-concrete objects (such as pencil marks), and with easily imagined objects in described situations. Ultimately he comes to a confident knowledge of $7 + 5 = 12$, a knowledge full of meaning because of its frequent verification. By this time, the difficult stages of learning will long since have been passed, and habituation occurs rapidly and easily.

It is impossible to illustrate at such length all the implications of the "meaning" theory for relating instructional pace to learning difficulty. It will be possible here only to refer to one type of change in practice which has possibilities not yet fully appreciated. This change is to "spread" instruction in various arithmetic topics over a wider span of the grades than is now the custom. Material progress along this line has been made in recent years, but much more can be done. To illustrate, many of the characteristics of common fractions are well within the intellectual grasp of primary grade children and are properly taught in these grades. Others of the easier learned aspects of fractions may well be "teased out" and taught through Grades 2 and 4, and the most difficult ones, when located, could be left for Grade 6, and even 7. There would seem to be little justification for the common instructional organization which concentrates so much of teaching of fractions into a single grade. The proposed changes would utilize the earlier years for a slow, painstaking development of the basic meanings of fractions. The child would thus be prepared to understand better the systematic treatment of fractions assigned to Grade 5 and the more difficult features of the topic reserved for Grades 6 and 7.

Experience in the attempt to make arithmetic meaningful to children may some day demonstrate the wisdom of "spreading" in a similar manner instruction on other topics. Thus, some of the simplest multiplications and division combinations may well be learned in Grades 1 and 2, and the most difficult of these combinations (with 8 and 9 as multiplier and divisor) may be postponed to Grade 4. The division of integers by a digit may be introduced through the long division form, which would be employed in Grade 4 to teach as many as possible of the difficult features of the process; two-digit divisors could then be withheld until Grade 5, and some of the most difficult (and most unusual) types of division, until Grade 6. Decimals, denominate numbers and measurement, and even per cent and ratio might readily be made more intelligible and significant by adapting instructional pace to learning rate through "spreading" the teaching of these topics. Not the least of the advantages of these changes is that "spreading" would help to remove a large part of the burden of uninspired "maintenance drill" which now seems to have made itself an integral part of arithmetic instruction.

(*c*) *Emphasis upon relationships.*[12] The third way in which arithmetic instruction according to the "meaning" theory helps to make number sensible is by emphasizing relationships within the subject. Five illustrations are all that can be offered at this point.

According to the drill theory "6 and 5 are 11" should not be taught in close temporal proximity to "7 and 4 are 11," for fear that children will use the one to solve the other instead of establishing independent bonds for the two. According to the "meaning" theory, children's recognition of the relation between the two statements not only is not harmful—it is a distinct gain. In fact, it could even be insisted that unless the relationships were understood neither fact would be adequately learned. After all, the number system *is* a system, a fact which for some curious reason is withheld from children when they study arithmetic. As a system it contains relationships and connections which, if mastered, should enable children to make progress much more readily in their learning.

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[12] Let the reader not be disturbed by what may seem to be the resurrection of a well-laid ghost. It is true that nearly a half-century ago (following Grube) the attempt to systematize arithmetic instruction came to grief and was abandoned in favor of what has become, in some instances, almost an absence of logical organization. The "meaning" theory is no revival of the Grube ideas. If the emphasis here given to meanings, understandings, and rationalizations seems to bear a close resemblance to discarded practices, a closer scrutiny will reveal the resemblance to be less real than fancied. The mere fact that failure attended one plan of teaching which, though it did aim at understanding, was nevertheless psychologically and socially unsound, is slight reason to disapprove all other such instructional plans.

It has been said that the number system is a system. Our number system is a decimal system; its unit is 10. That is to say, our system is organized around 10.[13] A second way, then, in which the "meaning" theory would emphasize relationships is to make much more use of the unit 10 than is common—for example, in teaching children the meaning of numbers above 10, in teaching them to read and write such numbers, and in teaching them the addition and subtraction combinations with sums and minuends of 11-18. The principle of adding and subtracting by endings and the procedure in higher-decade addition and subtraction would be far more intelligible to children if developed in terms of the basic unit 10. The unit 10 also could be employed not merely to explain "carrying" in addition and in subtraction but also to introduce some of the earlier types of multiplication and division. Decimal fractions, their notation, and operations with decimals should be much more easily understood if familiarity has been acquired with 10 as the unit in whole numbers.

A third illustration of the way in which the "meaning" theory emphasizes relationships is found in connection with the topics, common fractions, decimal fractions, and per cent. As these topics are now taught, they commonly impress the child as three essentially unlike mathematical forms, which can, by certain mechanical methods, be changed back and forth as required by the textbook. Actually these mathematical forms are but three different ways of expressing the same ideas. It may be inferred that they will be better understood if their relationships rather than their differences are stressed in teaching.

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[13] For a valuable and stimulating discussion of certain ones of the points made in this section see: Wheat, Harry C., *The Psychology of the Elementary School*, Silver Burdett Co., 1931, pp. 135-142, or, better yet, all of Chapter IV.

A fourth place in which the "meaning" theory requires attention to relationships is in the matter of the mathematical operations themselves—addition, subtraction, multiplication, and division. Each of these processes stands for a special type of relationship: in addition, that of "putting together"; in subtraction, that of "taking away"; etc. That children do not understand the meaning of the relationships thus contained in these operations is only too frequently demonstrated in problem-solving. Children add when they should subtract, multiply when they should add, and so on. There is little reason why they should not be expected to make these errors. What is done in instruction to forestall the errors? A child learns "2 'n 3 're 5" and is supposed to understand that he has added. He learns "8 take away 2 're 6" and is supposed to know that he has subtracted. The nature of the operation is hardly indicated by these verbal statements. Even if it were, the use would be associated only with the signs "add," "+," "sum," etc.; understanding would not be complete enough to set off the appropriate activity in verbal problems in which, in place of a few specific symbols, hundreds of language forms are employed to express it. If the operations are to be understood properly, the mathematical relationships for which the operations stand must be definitely taught.

The fifth illustration of relationships which would be stressed in instruction according to the "meaning" theory is that of the forms used in arithmetic—the column in addition and subtraction, the placement of partial products in multiplication, the method of writing the quotient figures in division with respect to the figures in the dividend, and so on. These forms are frequently taught as tools, only the mechanics of which need to be known; and yet each one of these forms contains its own logic which adapts it perfectly to its function. This purpose can be taught to children. If children understand it, they will have acquired another valuable relationship, namely, the relationship of form of expression to the thought to be expressed. The need of keeping numbers in their proper order place (1's at the right, 10's next, etc.) is usually demonstrated in the case of column addition, too often, however, with the hope merely of securing thereby a greater number of correct answers. The form used in multiplication can likewise be explained. Consider the series of teaching steps given on the following page.

Example: $\begin{array}{r} 42 \\ \times\ 2 \\ \hline \end{array}$

(1) 42 = 4 tens, 2 ones (2) $\begin{array}{r} 2 \\ \times\ 2 \\ \hline 4 \end{array}$ (3) $\begin{array}{r} 4 \text{ tens} \\ \times\ 2 \quad \\ \hline 8 \text{ tens} \end{array}$ *or* $\begin{array}{r} 40 \\ \times\ 2 \\ \hline 80 \end{array}$

(4) answer = 80 + 4 = 84 (5) $\begin{array}{r} 42 \\ \times\ 2 \\ \hline 4 \\ 80 \\ \hline 84 \end{array}$ (6) $\begin{array}{r} 42 \\ \times\ 2 \\ \hline 4 \\ 8\ \\ \hline 84 \end{array}$ (7) $\begin{array}{r} 42 \\ \times\ 2 \\ \hline 84 \end{array}$

Steps (1) to (4) offer no difficulty to the child, for he has long since learned the ideas and procedures involved. They are included in the presentation, however, because they review the multiplication of 10's and of 1's and, more important, they yield the answer of the new example *before* the new form is to be taught. Steps (5), (6), and (7) simply translate the known operations and answers into the new form, which then takes on the meaning which is associated with steps (1) to (4). Thereafter, the form is not a senseless device—it is an intelligible instrument for expressing a relation.

Development of quantitative thinking. According to the "meaning" theory the ultimate purpose of arithmetic instruction is the development of the ability to *think* in quantitative situations. The word "think" is used advisedly: the ability merely to perform certain operations mechanically and automatically is not enough. Children must be able to analyze real or described quantitative situations, to isolate and to treat adequately the arithmetical elements therein, and to make whatever adjustments are required by their solutions. When the purpose of arithmetic instruction is defined in the above terms, true arithmetical learning is seen to be a matter of growth which needs to be carefully checked, controlled, and guided at every stage. It cannot safely be presumed that children can themselves find and follow the most advantageous course of development. On the contrary, the responsibility for sound and economical growth rests squarely upon the teacher.

In meeting this responsibility the teacher is unwise who measures progress purely in terms of the rate and accuracy with which the child secures his answers. These are measures of efficiency alone, not of growth. It is possible for the child to furnish correct answers quickly, but to do so by undesirable processes. The true measure of status and of development is therefore to be found in the level of the thought process employed. If the teacher is to check, control, and direct growth, she must do so in terms of the child's methods of thinking. If the child tends to rest content with a type of process which is low in the scale of meaning, she will lead him to discover and adopt more mature processes. If she asks him to "explain" an exercise written on the blackboard or on his paper, she will not be satisfied merely to have him read what he has written; she will insist upon an interpretation and upon a defense of his solution. She will make the question, "Why did you do that?" her commonest one in the arithmetic period. Exposed repeatedly to this searching question, the child will come soon to appreciate arithmetic as a mode of precise thinking which derives its rules from the principles of the number system.

Possible criticism of the "meaning" theory. Those who subscribe to the "meaning" theory must expect criticism. The objections of the exponents of the drill theory and of the theory of incidental learning may be easily imagined. But there is another type of criticism which comes from a different quarter. To some writers in the field of arithmetic the "meaning" theory is but an attempt to restore dignity to the discredited faculty psychology and to revive the dead issue of formal discipline. These writers insist that the term "quantitative thinking" is but a vague if lovely phrase—gibberish—which, if it makes sense at all, is misleading in these practical days when teachers need to think concretely and specifically about arithmetic instruction.

There are two answers to this criticism. The first is to show that "quantitative thinking" is not pure fiction, but an actuality commonly within the experience of those who understand number—even of those who maintain that there is no such thing as "quantitative thinking." The lecturer who has strayed somewhat from his prepared remarks happens to draw out his watch. His response comes almost immediately: "Here! Here! I must hurry." An artificial analysis of his behavior would reveal the following reactions to have taken place: telling the time, noting the amount of time left in the hour, estimating the ratio of elapsed to available time, determining the amount of the lecture already given, the same for the amount yet to be given, computing the ratio between the two, comparing the two ratios, and arriving at a judgment.

Only, of course, the lecturer experiences nothing whatsoever of this sort. The numerous complicated computations isolated above never take place at all—certainly the lecturer is unaware of any such operations. Instead of making this analysis and this series of separate computations, the lecturer engages in a bit of instantaneous "quantitative thinking," and his adjustment following upon the thinking is as real and as adaptive as any other activity in his life. His behavior is inexplicable except as the result of a highly organized system of thought processes. It resembles the operation of a group of mechanical units only in that the individual does not *seem* to be actively the director of the behavior, as certainly he is, however. His behavior is purposive and extraordinarily intelligent. It would be impossible to an individual who had not developed the most mature types of quantitative thought processes. To say that the lecturer engaged in a skillful bit of "quantitative thinking" is, then, merely to *describe* his behavior—it is not to *explain* it; least of all is it to ascribe that behavior to the operation of some obscure "faculty." Mental organization does not imply "faculties."

The criticism of the "meaning" theory as requiring impossible mental feats and nonexistent mental faculties is, in the second place, founded upon an inadequate conception of the learning process and of transfer of training in particular. To set as the end of arithmetic instruction the development of the ability to think precisely in quantitative situations is not to call upon magic; it is simply to insist that the greatest possible advantage be taken of the capacity of mind to generalize. It is incorrect to say (though the statement is still frequently enough made) that experimental research has disproved the fact of transfer of experience. Research has done nothing of the kind. It has, however, demonstrated that if transfer is desired from one learning situation to another, then the training must be such as to assure transfer. No one who writes about "quantitative thinking" assumes that transfer is to be assured in any way except through arithmetic instruction designed to secure it. The nature of this instruction has already been described in the foregoing pages.

IV. CONCLUDING STATEMENT

The record of arithmetic in the school is an unenviable one. The position taken in this chapter is that the fault lies in the type of instruction generally given. Arithmetic instruction has for a number of years inclined much too far in the direction of the drill theory of teaching. The trend now seems to be in the direction of the incidental theory of instruction. While this change in instructional theory represents distinct improvement, it does not, for reasons given in the foregoing pages, promise the kind and amount of reform needed. An attempt has been made in this chapter[14] to outline a general shift in instructional emphasis and an altered view of the nature and purpose of arithmetical learning which may bring about the desired consequences. The basic tenet in the proposed instructional reorganization is to make arithmetic less a challenge to the pupil's memory and more a challenge to his intelligence.

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[14] Most of the illustrations of, and arguments for, the "meaning" theory have been drawn from the field of primary number. This fact should not, however, be interpreted to mean that the theory holds only for the first three grades. On the contrary, meaning affords the soundest foundation for arithmetical learning throughout the elementary school. Primary number has been most often cited for another reason. Almost everyone agrees that children in Grades 5, 6, 7, and 8 have to be able to "think" in arithmetic. That ability to "think" in these grades is conditioned by "thinking" in the primary grades is a fact which is much less commonly recognized. No one has shown how it is possible for children suddenly to become intelligent in upper-grade arithmetic when they have been allowed no exercise of intelligence in lower-grade arithmetic. In spite of the unreasonableness of such an expectation, primary number is taught as if skills acquired mechanically would later surely take on meaning, and verbalizations memorized unintelligently would later inevitably become well-rounded concepts. It is the thesis of the "meaning" theory that children must from the start see arithmetic as an intelligible system if they are ever to be intelligent in arithmetic. Hence, in this chapter, the implications of the "meaning" theory for the primary number have been especially stressed.

AN ANALYSIS OF INSTRUCTIONAL PRACTICES IN TYPICAL CLASSES IN SCHOOLS OF THE UNITED STATES

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ANYONE who has kept abreast of the rapidly growing body of research in arithmetic knows that many investigations have been made of such aspects of instruction as the grade placement of topics in courses of study, time allotments, pupil achievement as measured by standard tests, and the analysis of materials of instruction. The interesting thing is that there is very little information available as to actual teaching practices used by instructors in classrooms. There have been published brief reports of instruction in general in school surveys but practically none of these have emphasized arithmetic. It is the purpose of this chapter to present some of the results of a coöperative survey of actual teaching practices in elementary schools in which cities in all parts of the country participated. It is hoped that the information in the following pages will serve as a basis of action for those who are interested in the problem of improving arithmetic instruction.

The procedure used. The procedure used in making the survey was very simple. First, a blank was developed (see Figure 1) containing a series of items grouped under various major headings. Each of the items on the blank represents either a judgment of an observer of a lesson or a specific activity which might be observed as the lesson proceeded. The list grew out of many observations of lessons in arithmetic by students in courses in the supervision and teaching of arithmetic at the University of Minnesota. By means of this blank it was a very simple matter for an observer to record various types of information concerning a class, which would be useful in analyzing and evaluating the instruction. By consolidating the information for a large number of lessons a vivid picture of current practice would be readily available.

Letters were written to superintendents of the larger school systems throughout the country asking them to assist in the survey by naming principals or supervisors who would be willing to coöperate by reporting observations for two classes apiece. A copy of the blank was included in each letter. The response was prompt and generous. Requests for approximately 1,200 blanks were received.

Full directions for making the survey were then sent with the blanks. The classes were to be observed, if possible, in the first two weeks in May, 1933. No special preparation for the observation was to be made by the teacher. In so far as possible typical day-to-day lessons were to be reported. The observer was to be in the class throughout the whole lesson. The blank to be used for recording the observation was not to be discussed beforehand with the teacher.

Data regarding the classes. In Table I are given various data concerning the classes included in this report.

TABLE I

GENERAL DATA REGARDING CLASSES INCLUDED IN THE REPORT

Grade	Number of Classes	Median Class Size	Median Length of Period	Median Experience of Teachers (Years)
Grade 4	153	43	40	9
Grade 5	170	39	40	16
Grade 6	182	40	40	16
All classes	505	40	40	14

In all there were reports for 505 observed lessons, the data from which were received in time to be included in the tabulations (June 1, 1933). There were 153 classes in Grade 4, 170 classes in Grade 5, and 182 classes in Grade 6. These classes were divided between traditional schools and platoon schools in the ratio of approximately six to one. Because of the relatively small number of classes of the platoon-school type, data for all classes were consolidated. Results from very few experimental classes were received. Geographically the classes were distributed throughout the country from New England to California. The cities that assisted in the survey were all large. No data are included for small towns and rural communities. The median length of the class period in all grades was 40 minutes. A few classes had periods of 60 minutes

FIGURE 1

SURVEY OF SELECTED INSTRUCTIONAL PRACTICES IN OBSERVED LESSONS IN ARITHMETIC IN GRADES 4, 5, AND 6 CONDUCTED UNDER THE AUSPICES OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

School.............................. *City* *State*.............. *Grade*........

Length of Period...................... *Time of Day*..................... *Number of Pupils*........

Experience of Teachers (in Years)...................... *Type of School*
- Platoon ....
- Traditional ....
- Experimental ....
- Other ....

TRAINING: Grade 9, 10, 11, 12 Normal or Teachers College 1, 2, 3, 4 University or College 1, 2, 3, 4.
(Encircle highest year attended)

DIRECTIONS: This blank is to be used to record certain observed facts in *one* typical lesson in arithmetic, in a room having only one full class in it. Split classes should not be included. The observer will check the items below applying to this *one* lesson alone. Space is provided in each group of items for additional facts that may appear to be vital.

1. The apparent *major* objective of the lesson. (Check not more than two, preferably only one unless two are very evident.)

....a. To develop skill in computation.
....b. To develop skill in solving problems stressing processes rather than social applications of number.
....c. To develop an understanding of the *social* applications and uses of number in life.
....d. To develop interest in number through various types of projects, creative activities, and the like, planned by the pupils, under the guidance of the teacher.
....e. Others, such as ....

2. Types of instructional activities occurring. (Check all those observed.)

....a. Development of new process by direct teaching
....b. Practice on the new process
....c. Review of previous work through discussion and questions
....d. Formal written drill on previously acquired skills
.... (1) With standard drill materials
.... (2) With teacher prepared materials
....e. Games
....f. Test not including practice tests listed in d.
.... (1) With test prepared by the teacher
.... (2) With a standardized test in arithmetic
....g. Pupils give original problems illustrating topic under consideration
....h. Teaching pupils how to solve problems using prepared problem solving helps
....i. Practice in solving problems dealing with social applications of number
....j. Practice solving problems to illustrate some computational process
....k. Discussion of historical aspects of number
....l. Discussion of present day social applications and uses of number
....m. Planning and executing class project involving practical application of number
....n. Reports on assigned topics, assigned reading, etc.
....o. Worthwhile voluntary independent contributions made by individual pupils
....p. Dramatizations of applications of number
....q. Systematic diagnostic work with individuals by the teacher
....r. Pupils taught how to diagnose their own difficulties
....s. Pupils diagnose their own difficulties
....t. Systematic remedial work adapted to individual needs
....u. Oral practice exercises for speed work
....v. Independent group work by some pupils
....w. Completely individualized work (Winnetka plan)
....x. Presentation of uses of number in other subjects, as geography, health, etc.
....y. Construction of graphs and other types of geometric design
....z. Others, such as ....

FIGURE 1 (*Continued*)

. Instructional materials used. (Check all those used.)

a. Books

.... (1) No books used
.... (2) Basic text in hands of pupils
.... (3) Supplementary textbooks
.... (4) Reference books, encyclopedias, etc.
.... (5) Pamphlets, bulletins, magazines, etc.
.... (6) Selections found in readers, geography texts, history texts, etc.
.... (7) Others, such as ....

b. Practice exercises

.... (1) Exercises in textbook
.... (2) Standardized drill cards adapted for individual progress
.... (3) Unstandardized materials on cards prepared by the teacher
.... (4) Mimeographed materials
.... (5) Workbooks
.... (6) Materials on blackboard to be copied by pupils
.... (7) Dictated materials to be copied by pupils
.... (8) Problems or examples given orally to be solved mentally
.... (9) Flash cards
.... (10) Others, such as ....

c. Other equipment

.... (1) Blackboard used by teacher
.... (2) Blackboard used by pupils
.... (3) Slides, films, etc.
.... (4) Class progress graph (in use or on wall)
.... (5) Individual progress graph
.... (6) Charts, diagrams, pictures, etc., not in textbook
.... (7) Objects, such as cubes, measures, sticks, rulers, instruments, etc.
.... (8) Illustrative materials collected from the community
.... (9) Bulletin board display of current applications of number
.... (10) Prepared exhibits of materials supplied by commercial houses
.... (11) Neatness scales to set standards
.... (12) Others, such as ....

4. The organization of the room for work.

....(a) The entire class does the same work on processes
....(b) The entire class does the same work on problems
....(c) Pupils are divided into two or more groups according to progress made
....(d) There are independent groups working on various group projects
....(e) There is completely individualized instruction on number processes
....(f) Others, such as ....

5. Basis of the class work. (Check one.)

....(a) Teacher directed activities limited almost wholly to the organization and content of a single textbook or of the drill exercises
....(b) Variety of materials is introduced by the teacher to supplement the textbook to enrich instruction and to develop interest by the pupils
....(c) Work is organized in large units of subject matter devised by the teacher and executed by the pupils (Contracts, Morrison units, etc.)
....(d) The class work is organized in the form of activities planned and executed by the pupils under the guidance of teacher. (Projects, creative activities, etc.)

Person making the observation..

Position..................................

Please return all blanks when completed to

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in length, while a few were less than 30 minutes. The median size of all classes was 40 pupils. Classes for Grade 4 were slightly larger than for the other two grades. The median experience of the teachers for all grades combined was 14 years. Teachers in Grade 4 had a median of only 9 years' experience, 5 years less than for the teachers in Grades 5 and 6. The teachers whose work was observed by those who filled in the report blank were undoubtedly a selected group of mature, experienced teachers of ability. Table II shows the training of the teachers. Data on training were given on only 450 of the report blanks. It can be seen from Table II that very few of the teachers had had less than two years of college work, while a large number had had as much as four or even five years of college education, either in normal schools or in universities, or both. This level of education is on the whole considerably higher than is usually found in a group of five hundred elementary school teachers. It is, therefore, clear that the summaries that follow are based on consolidations of observed lessons taught by mature, selected individuals of training considerably above the average.

TABLE II

YEARS OF TRAINING OF TEACHERS OF ARITHMETIC

(450 blanks included data)

Place of Training	Number of Years				
	1	2	3	4	Total
Teachers colleges	20	160	76	34	290
Universities	6	9	19	58	92
Combinations	68 (39 of less than 5 years; 29 of 5 years or more)				

Apparent major objectives of lessons observed. The first item that the observer was asked to record about a lesson was its apparent major objective. Extensive preliminary observations of typical lessons had revealed the fact that these objectives could conveniently be stated under four heads, as given in Table III.

In Table III are given data showing how frequently each objective was checked as an apparent major objective of a class. The directions to observers were that not more than two objectives were to be checked for a lesson, preferably only one, unless two were very evident. The purpose of this plan was to try to get a picture

TABLE III

Number of Times Each Objective Was Reported[*] as The Apparent Major Objective of the Lesson

Objective	Grade 4		Grade 5		Grade 6		All	% of Total
	No.	%	No.	%	No.	%		
a. To develop skill in computation	86	56	96	56	87	48	269	53
b. To develop skill in solving problems stressing processes rather than social applications of number	56	37	74	44	76	42	206	41
c. To develop an understanding of the social applications and uses of number in life	24	16	33	19	65	36	122	24
d. To develop interest in number through various types of projects, creative activities, and the like, planned by the pupils under the guidance of the teacher	5	3	8	5	7	4	20	4
e. Others, such as ——	9	6	11	6	4	2	24	5
Number of classes	153		170		182		505	

[*] *Note:* The directions for reporting apparent major objectives of lessons were: (Check not more than two, preferably only one unless two are very evident.)

of what in the judgment of the observer was the best statement of the major objective of the lesson. Data are given for each grade, and for all grades combined.

It is obvious from Table III that the apparent major objectives of the great majority of the observed lessons were to develop skill in computation and in solving verbal problems, stressing the various processes rather than the social applications. In 53% of all classes observed, "To develop skill in computation" was checked as the major objective and in 41% of all classes, "To develop skill in solving problems stressing processes rather than social application of numbers" was checked as the major objective. In only 24% of the classes was the third objective, "To develop an understanding of the social applications and uses of number in life," judged by the observer to be the major one. In only 4% of the classes was the major objective the development of interest in number through various types of projects, creative activities, and the like, planned by the pupils under the guidance of the teacher. In about 30% of the cases more than one objective was checked, usually a combination of a and b, or a and c. It is worth pointing out that the emphasis on the objective, "The understanding of social application and uses of number," increases considerably from Grade 4 to Grade 6, while the emphasis on developing skill in computation is less in Grade 6 than in Grade 4. In only 24 cases, or 4%, of the total did observers list objectives other than those given; most of them were in fact only minor variations of the four used in the blank.

The evaluation of the facts presented in Table III depends largely on one's conception of what the functions of instruction in arithmetic are. Elsewhere an attempt has been made to state what in the opinion of progressive curriculum workers in arithmetic these functions are or should be.[1] Briefly stated, they are as follows: (1) the computational function, which includes instruction in the arithmetic processes useful in everyday life; (2) the informational function, which emphasizes the belief that instruction in arithmetic should include the presentation of significant information concerning the development and application of number in the affairs of daily life; (3) the sociological function which emphasizes the sociology of number and the understanding of the ways in which number has facilitated social intercoöperation; and (4) the psychological function which emphasizes what Dr. Judd has well described as "number as a mode of thought," that is, the ability to think precisely and accurately by means of quantitative techniques that have been or are being developed by man.

In a well-balanced course in arithmetic each of these four functions should be stressed. In the opinion of many competent persons the informational, sociological, and psychological functions are of greater significance and value than the computational. When teachers emphasize the development of skill in computation they tend to neglect the other functions. Judging from the returns received from the 505 classes taught by selected teachers it is obvious that in this country teachers have as their major objective the development of skill in computation. This conclusion is similar to that of Steel who found that teachers in Grades 4 to 6 in the schools of Minnesota devoted approximately 84% of the class time to drill on computational processes. The result must undoubtedly be that arithmetic is in danger of becoming a formal, devitalized subject instead of the rich social study it could so easily be.

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[1] See the "Critique" in *The Twenty Ninth Yearbook of the National Society on the Study of Education.* Public School Publishing Company, Bloomington, Ill., 1930.

The explanation for this emphasis on skill in computation is quite simple. At the present time the achievement of pupils in arithmetic is measured by the scores they make on the various standard tests so widely used throughout the country. Practically all of these tests contain only examples to be solved by computation. Since in many quarters undue weight is given to the results of these tests, teachers stress the work which will enable their pupils to make high test scores, regardless of the social value of the processes presented. Likewise our courses of study and most of our textbooks stress the computational aspects of the subject. The solution of this difficulty does not necessitate the elimination of all tests but rather the development of new tests which will enable the teacher to determine the extent to which the pupils have mastered the essentials included under the other three functions of the subject.[2] Tests which measure the extent to which all of the desired objectives are being achieved are one of the most valuable professional tools the teacher has available.

The organization of the classwork. Class work in any subject may be narrow and limited to a textbook, or it may be organized into large, rich units or activities. In recent years there has been much discussion of the theory of the activity curriculum and methods of organizing the work in various subjects into units, such as contracts, understanding units, Morrison units, and the like. In Table IV data are presented which show what the basis of the class work in arithmetic in the 505 observed classes was in each case judged by the observer to be.

The data in Table IV show conclusively how great an influence the textbook has on the kind of instruction pupils receive in arithmetic. In Grade 4 in 53 classes, or 34% of the total for the grade, the basis of the class work was teacher-directed activities limited almost wholly to the organization and content of a single textbook or of the drill exercises used; in Grade 5 the per cent of all classes having the textbook as the basis of class work was 41; in Grade 6 the per cent was 41; the textbook or drill materials was the basis of the class work in 39% of all classes. When modern, scientifically constructed textbooks which present a balanced treatment of the four functions of arithmetic are being used, these facts are not

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[2] The writer has made a preliminary attempt to measure these functions in a series of tests, entitled Analytical Scales of Attainment in Arithmetic, published by the Educational Test Bureau, Inc., Minneapolis, Minn.

TABLE IV

BASIS OF THE CLASS WORK*

Class Work	Grade						All	% of All
	4		5		6			
	No.	%	No.	%	No.	%		
a. Teacher directed activities limited wholly to the organization and content of a single textbook or of the drill exercises	53	34	69	41	71	41	193	39
b. Variety of materials is introduced by the teacher to supplement the textbook to enrich instruction and to develop interest by the pupils	68	44	78	46	87	48	233	47
c. Work is organized in large units of subject matter devised by the teacher and executed by the pupils (Contracts, Morrison units, etc.)	1	1	6	4	4	2	11	2
d. The class work is organized in the form of activities planned and executed by the pupils under the guidance of teacher. (Projects, creative activities, etc.)	2	1	9	5	7	4	18	3
e. Others	2	1	1	1	1	1	4	1
f. Not reported	27	18	7	4	12	7	46	8
Number of classes	153		170		182		505	

* *Note:* Observers were asked to check one item.

alarming. However, when one realizes the limitations of some of the older books that are now in the hands of the pupils, it is apparent that a very meager type of subject matter is taught in many arithmetic classes.

That teachers recognize the limitations of instructional materials and realize the necessity of methods of supplementing the textbooks as a basis for enriching the class work is shown by the data in Table IV. In 47% of all classes the teacher introduced a variety of materials to supplement the textbook so as to enrich the instruction and to develop the pupil's interest. This is a very satisfying situation.

In very few classes was there evidence that the new progressive theories of education have influenced instruction in arithmetic. In only 2% of all classes was the work organized in large subject-matter units. In only four classes was the class work organized in the form of activities, such as projects, excursions, creative activities, and the like. This condition is undoubtedly largely due to the fact that instruction in arithmetic is at present limited chiefly to the computational function and since this results in a great amount of formal drill work very little use is made of the various types of socializing experiences which have done so much to vitalize instruction in the other subjects of the curriculum. That much can be done to enrich the teaching of arithmetic is being demonstrated in many schools, some of which have published descriptions of the units[3] they have developed.

Provision for individual differences. In these days when educational literature abounds with evidence of the fact of individual differences it is interesting to note what is being done by teachers of arithmetic to adjust instruction to the differences in ability, rate of learning, and progress of pupils in their classes. All pupils in a class may be required to do the same work, or they may be divided into groups according to their needs; or the work may be individualized, as is done in Winnetka, Ill., and Detroit, Mich. Data on these points are given in Table V.

TABLE V

ORGANIZATION OF THE ROOM FOR WORK

Organization	Grade						All	% of
	4		5		6			
	No.	%	No.	%	No.	%		
a. The entire class does the same work on processes	69	45	79	46	77	42	225	45
b. The entire class does the same work on problems	43	28	53	31	66	36	162	32
c. Pupils are divided into two or more groups according to progress made	64	42	96	56	93	51	253	50
d. There are independent groups working on various group projects	14	9	17	10	22	12	53	10
e. There is completely individualized instruction on number processes	9	6	12	7	8	4	29	6
f. Others, such as ——	6	4	5	3	7	3	18	3
Number of classes	153		170		182		505	

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[3] See, for example, the bulletin, *Arithmetic Activities*, by Alma B. Caldwell, published by the Board of Education, Cleveland, Ohio; also, the *South Dakota Course of Study in Arithmetic*, published by State Department of Education, Pierre, S. D.

Data in Table V show that in approximately 45% of all classes the entire class did the same work on processes and in 32% the entire class did the same work on problems. There was not much variation in the per cents in each grade. In these classes the fact of individual differences was apparently ignored. On the other hand, in approximately 50% of the classes the pupils were divided into two or more groups according to the progress made, while in 10% of all classes there were independent groups working on various group projects. In only 6% of all classes was the work on number processes completely individualized. In view of the excellent types of instructional materials in arithmetic intended for the purpose of individualizing instruction in arithmetic processes that can easily be secured the small proportion of classes which used this fact is surprising. The fact that in so many classes pupils were divided into groups according to ability shows that definite attacks are being made on the problem of individual differences. The procedures used vary from city to city. As far as increasing the efficacy of drill work on computation is concerned, the solution seems clearly to be the development of a type of class organization which completely individualizes the work. Present practices in many classes should be radically changed. As far as the socializing aspects of the subject as emphasized by the informational, sociological, and psychological functions are concerned, it seems reasonable to maintain that a wide variety of types of activities in which the pupils may all work as a single class group on one problem, or in smaller groups on various subtopics is desirable. These activities could be quite similar to those that have been found worthwhile in the social studies, namely, discussions of present-day problems, assignments of special topics for investigation and report, continuation projects, excursions, dramatizations, and other types of socializing group experiences.

Types of instructional activities. The types of instructional activities engaged in by the class give a good picture of the methods and procedures that are being used by the teacher to vitalize the subject, or to devitalize it, as the case may be. In order to secure data on this point a list of twenty-seven widely varied types of activities that might take place in arithmetic classes was prepared. Observers were asked to check on the list those activities that actually occurred during the lessons observed. The consolidated data for each grade and for all combined are given in Table VI.

TABLE VI

TYPES OF INSTRUCTIONAL ACTIVITIES IN OBSERVED CLASSES IN ORDER OF FREQUENCY

Instructional Activity	Grade			All	% of All
	4	5	6		
	No.	No.	No.		
a. Review of previous work through discussion and questions	65	92	110	267	53
b. Practice on the new process	71	80	70	221	44
c. Development of new process by direct teaching	60	72	66	198	39
d. Practice solving problems to illustrate some computational process	45	69	71	185	37
e. Formal written drill on previously acquired skills with teacher prepared materials	49	60	59	168	34
f. Pupils taught how to diagnose their own difficulties	48	60	58	166	33
g. Systematic remedial work adapted to individual needs	47	54	64	165	33
h. Pupils diagnose their own difficulties	44	52	68	164	33
i. Practice in solving problems dealing with social applications of number	39	49	71	159	32
j. Systematic diagnostic work with individuals by teacher	43	54	47	144	29
k. Oral practice exercises for speed work	39	52	40	131	26
l. Formal written drill on previously acquired skills	31	41	43	115	23
m. Independent group work by some pupils	28	38	40	106	21
n. Worthwhile voluntary independent contributions made by individual pupils	18	31	41	90	18
o. Teaching pupils how to solve problems using prepared problem solving helps	27	30	28	85	17
p. Pupils give original problems illustrating topic under consideration	17	40	27	84	17
q. Discussion of present day social applications and uses of number	19	18	45	82	16
r. Formal written drill on previously acquired skills with standard drill materials	19	20	26	62	12
s. Test prepared by the teacher	9	19	18	46	9
t. Games	9	14	4	27	5
u. Dramatizations of applications of number	9	6	11	26	5
v. Presentation of uses of number in other subjects, as geography, health, etc.	3	4	13	20	4
w. Construction of graphs and other types of geometric design	2	10	8	20	4
x. A standardized test in arithmetic	3	6	7	16	3
y. Planning and executing class project involving practical application of number	3	3	12	18	3
z. Reports on assigned topics, assigned reading, etc.	0	2	7	9	2
a′ Discussion of historical aspects of number	1	0	4	5	1
b′ Completely individualized work (Winnetka plan)	1	1	1	3	1
c′ Others, such as ——	5	14	16	35	7
Number of classes	153	170	182	505	

In Table VI the activities are listed in the order of the frequency with which they were checked for all classes combined. The activity given first in the table occurred most frequently, the one given last occurred least often.

The data in Table VI merely show in detail what has been previously reported. The five activities checked most frequently were: (1) review of previous work through discussion and questions, 53% of all classes; (2) practice on new process, 44% of all classes; (3) development of a new process by direct teaching, 39% of all classes; (4) practice in solving problems to illustrate some computational process, 37% of all classes; and (5) formal written drill on previously acquired skills with teacher-prepared materials, 34% of all classes. These data show the extent to which the computational function was stressed in the classes included in this report. In view of the fact that the number of lessons is quite large, that no special preparation was made by the teachers for the observations, and that the lessons were to be typical ones, the conclusions based on another similar set of observations would probably not differ widely from those herein presented.

Almost all types of activities that would be expected to occur in classes in which an effort was being made to socialize and to vitalize the work in arithmetic appear far down in the list in Table VI. For example, dramatization of applications of number ranks twenty-first, presentation of uses of number in other subjects ranks twenty-second, planning and executing class project involving practical application of number ranks twenty-fourth, reports on assigned topics, assigned readings, etc., ranks twenty-fifth, discussion of historical aspects of number ranks twenty-sixth. Not only do these activities rank low in the list but they are also reported as occurring in very few classes. It is interesting to note that in approximately one-third of all classes diagnostic and remedial work of various kinds was being done. This shows that teachers are making definite efforts to apply the techniques that have been devised by various investigators to aid teachers to determine the reasons why many pupils fail to make satisfactory progress in arithmetic. This is undoubtedly one factor that has led to the grouping of pupils according to their needs which has been shown to be characteristic of approximately half or the classes included in this report.

Instructional materials used. Another angle of the teaching of arithmetic is revealed by an analysis of the various kinds of instructional materials that were used by the teachers. At the present time one finds a wide variety of materials available—textbooks, workbooks, standard drill cards, such objects as cubes, measures, and the like—to aid in making the work concrete, progress graphs, and printed bulletins dealing with special topics. Likewise teachers often find it helpful to bring in displays of local materials, commercial exhibits, and other items that illustrate local applications of number. Data on the extent to which the different kinds of instructional materials were used in the classes included in this report are given in Table VII.

The data in Table VII are grouped under the three heads, books, practice exercises, and other equipment. The number of times each type of material was reported for each grade and for all grades combined is given. In 257 classes, or 51% of the total number, a basic textbook was used by the pupils. This shows the extent to which teachers rely on textbooks in teaching arithmetic. On the other hand, in 181 classes, or 36% of the total, no books were used at all. Are present-day textbooks inadequate? Where do pupils find the materials for the lesson? Under the heading "practice exercises" we find that in 197 classes, or 37% of the total, the pupils copied materials placed on the blackboard; in 15% of the classes the pupils copied dictated materials. Both of these practices are very inefficient. The danger of error in copying long examples, the difficulty of seeing materials written on the blackboard, and the time consumed in copying are all strong arguments against these practices. Furthermore, it is practically impossible to provide for individual differences when the teacher attempts to place the exercises for the lesson on the blackboard. Invariably all pupils do the same work. The teacher simply cannot prepare assignments for the various individuals in the class. This condition is not helped much when practice exercises given in textbooks are used, as is done in 39% of all classes. Textbooks as they are now constructed usually make no provision for assignments adapted to the needs of individual pupils, although an examination of some of the more recent textbooks shows that definite steps are being taken to improve this condition by means of inventory tests to locate specific needs of pupils, diagnostic tests to determine specific weaknesses, study helps for self-help by pupils, and practice tests for remedying deficiencies.

At present probably the most effective type of drill materials

TABLE VII

Number of Times Each Type of Instructional Materials Was Observed

Instructional Material	Grade 4 No.	Grade 5 No.	Grade 6 No.	All	% of All
a. Books					
1. No books used	48	62	71	181	36
2. Basic text in hands of the pupils	71	86	100	257	51
3. Supplementary textbooks	18	13	11	42	8
4. Reference books, encyclopedias, etc.	0	0	4	4	1
5. Pamphlets, bulletins, magazines, etc.	1	4	10	15	3
6. Selections found in readers, geography texts, history texts, etc.	0	4	9	13	2
7. Others, such as ——	7	14	9	30	6
b. Practice exercises					
1. Exercises in textbook	53	70	75	198	39
2. Standardized drill cards adapted for individualized progress	16	13	14	43	8
3. Unstandardized materials on cards prepared by teacher	20	28	11	59	12
4. Mimeographed materials	20	38	26	84	17
5. Workbooks	22	15	22	59	12
6. Materials on blackboard to be copied by pupils	54	74	68	196	37
7. Dictated materials to be copied by pupils	17	32	28	77	15
8. Problems or examples given orally to be solved mentally	26	44	33	103	21
9. Flash cards	18	11	12	41	8
10. Others, such as ——	6	9	8	23	5
c. Other equipment					
1. Blackboard used by teacher	107	138	137	382	76
2. Blackboard used by pupils	96	131	133	360	72
3. Slides, films, etc.	0	0	1	1	1
4. Class progress graph (in use or on wall)	29	27	40	96	19
5. Individual progress graph	26	31	47	104	21
6. Charts, diagrams, pictures, etc., not in textbook	22	20	24	66	13
7. Objects, such as cubes, measures, sticks, rulers, instruments, etc.	16	15	22	53	11
8. Illustrative materials collected from the community	4	4	10	18	4
9. Bulletin board display of current applications of number	2	6	7	15	3
10. Prepared exhibits of material supplied by commercial houses	0	0	2	2	1
11. Neatness scales to set standards	6	4	1	11	2
12. Others, such as ——	3	5	7	15	3
Number of classes	153	170	182	505	

is to be found in the standardized drill cards or workbooks which make possible individualized progress. Table VII shows that in only 8% of all classes were such drill cards used and that in only 12% were workbooks used. In 12% of all classes the pupils used unstandardized drill materials on cards prepared by the teacher. When one appreciates the difficulties involved in preparing adequate standardized drill exercises, one wonders whether or not these exercises prepared by the busy teacher are efficiently constructed. It must be recognized that the picture here presented is undoubtedly affected to some extent by the lack of funds available for the purchase of important and necessary kinds of instructional materials.

Because of the need of supplementing textbooks teachers also make extensive use of the blackboard. This is shown by the fact that in 76% of the classes the teachers used the blackboard. In 72% of the classes pupils used the blackboard for part of the class work. In 17% of all classes mimeographed materials of various kinds were used. Lantern slides which enable the teacher to project on a screen various types of instructional materials, such as explanations, illustrations, or drill exercises, were used in only one of the 505 classes observed.

Methods of enriching and vitalizing instruction. Instructional materials of various kinds are used to enrich and vitalize class work. Table VII shows the extent to which they were used by the teachers in this survey. Under the heading of books we find that reference books, encyclopedias, etc., were used in less than 1% of the classes; pamphlets, bulletins, magazines, etc., which abound in illustrative materials in only 3% of the classes; and materials in readers, geography, and history textbooks in only 2% of all classes.

Here we have evidence that teachers of arithmetic do not regard it as their function to enrich and to socialize the subject. It is, of course, true that number is used incidentally in some of the other subjects, such as geography, history, and reading. It should be emphasized, on the other hand, that arithmetic itself offers so many possibilities of enrichment that definite provision for such enrichment should be made in the arithmetic class. There is no reason at all why the work in arithmetic should be limited to the computational function. At the present time textbooks do stress this function. One way to overcome this deficiency would be to supplement them with arithmetic readers containing informational material related to the topics being studied, short pamphlets on special topics, such as the inexpensive series of booklets on the "Achievement of Civilization," prepared by the American Council on Education, references to current periodical literature, and the like. Many of the lessons in the Teachers' Lesson Unit Series, edited by Professor W. A. McCall, contain excellent descriptions of enriched units in arithmetic emphasizing its social values. They usually give numerous references to suitable supplementary materials in books on the child level. Many of the reading textbooks contain interesting accounts of the history of time-pieces, the development of various kinds of money, the story of how people earn money, measurement, and other topics which can easily be integrated with the work in arithmetic. The maps, charts, tables, graphs, and other types of quantitative materials in social studies textbooks offer abundant opportunity for rich, meaningful instructional units in arithmetic. Instead of limiting the study of number to computation every effort should be made to vitalize it by giving the pupil the opportunity to discover the many ways in which number has enabled man to deal efficiently and simply with the many aspects of the environment which must be dealt with quantitatively if they are to be understood.

Use of progress graphs. Experiments on the motivation of learning have conclusively demonstrated the value of class or individual progress charts which enable the pupil to determine his progress from time to time. The pupil is thereby stimulated to work for his own improvement. It is interesting to note that in 19% of all classes class progress graphs were in use or on the wall; in 21% of the classes individual progress graphs were used. They should be used in all classes. There are available several series of standardized tests which enable the teacher to measure pupil progress at regular intervals throughout the year. The information supplied by the results of these tests makes it possible for the teacher to adjust her instruction to the needs of the pupils in an intelligent manner.

Use of illustrative materials. The more use the teacher can make of various kinds of concrete illustrative material, the richer and more vital the subject matter being studied is likely to be. In 53 classes, or 11% of all, such objects as cubes, measures, or instruments were used; in only 4% of the classes were illustrative materials collected from the community used; in only 3% of the classes were bulletin-board displays of current applications of number used; in only two classes was any use made of prepared exhibits of materials supplied by commercial houses. It seems obvious that much more use can be made of such types of illustrative materials than was made by teachers of the classes included in this survey.

Felt needs of teachers compared with current practices. It is very helpful to compare the inadequacies of arithmetic instruction as revealed by a survey of teaching practices with the points on which teachers feel the need of help. This is possible by using the results of an earlier survey[4] involving a large number of teachers of Grades 3 to 6 in schools of the Middle West, few of whom are included in the present survey. These teachers were asked to indicate on a check list of 128 items related to the teaching of arithmetic the degree to which they felt the need of assistance on the several items included in the list. The following twenty-five items include those on which the highest per cents of teachers indicated a felt need of help. It will be observed that the activities involved in these items correspond quite closely to the inadequacies of instruction revealed by the present survey.

1. Showing how number has aided in the systematizing of the quantitative aspects of the environment.
2. Knowing the historical development and significance of the important applications of number.
3. Using excursions, projects, and exhibits.
4. Showing the social significance of quantitative concepts.
5. Providing opportunities for exploring topics of special interest.
6. Keeping notebooks on topics being studied.
7. Assigning topics for special reports.
8. Providing for group or committee work on special topics.
9. Providing opportunities for pupils to assist in the organizing and planning of class activities and discussions.
10. Assigning special topics for independent research.
11. Organizing the aspects of the school bank, milk supply, and similar activities in which arithmetic is used in a practical way so that pupils participate in various phases of the activities.
12. Arousing interest and developing the habit of extensive reading on topics being considered by the class, such as banking, the history of number, etc.
13. Developing an appreciation of the social significance of quantitative relations.

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[4] Brueckner, L. J., White, L., and Dickeman, F., *A Curriculum Study in Teacher Training in Arithmetic*. University of Minnesota Press, 1932.

14. Training pupils in the use of reference materials.
15. Securing a learning situation in which motives of a relatively high order are present.
16. Providing intercorrelations between subjects in situations in which number functions.
17. Preparing informal diagnostic exercises.
18. Providing for variations in rates of pupil progress.
19. Using a socialized form of recitation.
20. Utilizing provisions that enable the teacher to secure test materials.
21. Filing informal tests for future use.
22. Using the school equipment such as mimeograph, etc., in preparing tests.
23. Coöperating with other teachers in the preparation of tests.
24. Preparing improved types of supplementary materials.
25. Reading new scientific contributions on aspects of arithmetic instruction.

Conclusions

It is recognized that the results of this survey are based on only a small sampling of classes. They were carefully selected, however, and probably the work in them represents a cross section of what may be regarded as better practices in the teaching of arithmetic in this country. It is suggested that similar surveys should be conducted in various schools to get a more complete picture of what local practices are. Such information could well be made the basis of a direct attack on the improvement of the teaching of arithmetic. It might also be one source of suggestions as to how our present courses in normal schools in the teaching of arithmetic might be revised in order to adapt them more closely to conditions as they exist in our schools. Certainly they should make some provision for remedying conditions which have resulted in the relatively narrow type of instruction which now so commonly characterizes instruction in arithmetic.

INFORMATIONAL ARITHMETIC

By B. R. BUCKINGHAM

In this chapter the term "informational arithmetic" will be contrasted with computational arithmetic. It will be taken to refer to understanding, interpretation, and use rather than to processes and skills. Interpreted in this way, it involves partly a difference in subject matter, partly a difference in method, and altogether a difference in purpose. It is not always easy or practicable to determine whether a given item in the course of study, the textbook, or the classroom procedure is informational or whether it is computational. Of course, extremes are readily identified. Drill in column addition or in long division or in finding interest at 6% is clearly computational. Instruction in our number system or in the meaning of a fraction or in the principles of investment is clearly informational. In many instances, however, a decision merely on the basis of subject matter is impossible. Much depends upon the way in which the subject matter is approached and the outcome which is sought in handling it.

It is also true that each of the two types of arithmetic serves the other. Practice in computation serves the purposes of information, for example, through the gradual engendering of a sense of the meaning of numbers. It does not do this for all pupils. Even those pupils who do grasp meanings in this way are likely to do so rather superficially. Nevertheless, it is a matter of common observation that number sense of some sort often arises from what appears to be the mere manipulation of digits.

On the other hand it is equally certain that informational arithmetic serves the purposes of calculation. Strong and vivid number concepts are a safeguard against absurdities in computation, while a vital sense of the meaning of each number process goes far to assure its successful execution.

Too much of the literature regarding arithmetic has exalted computation. The purpose of an arithmetic course has been taken to be the speed and accuracy with which one can add, subtract, multiply, and divide whole numbers and fractions and use the speed and accuracy thus attained in certain business applications. Endless articles and books have been written in accordance with this computational purpose. The analysis of processes, the identification of types of errors, the maintenance of skills, the distribution of practice —these and scores of other topics and expressions constitute the chapter titles and topical headings of these books and articles. Games, charts, flash cards, tests, practice pads, and remedial instruction are all pretty much in the service of computation. The writer would be the last to say that computation is not important. An arithmetic course which does not have as a very definite objective the development of ability to compute with accuracy and facility is wholly inadequate. However, the curriculum which tries to accomplish nothing but accuracy and facility in figuring is woefully one-sided. Moreover, a curriculum which sets out to do that and that only will not even serve its own purpose satisfactorily. The child who merely puts down 2 and carries 1, as a dog does a trick, who borrows merely according to rule, who inverts the divisor because the teacher says so, who places the decimal point in multiplication in unquestioning compliance with a device in the book, will not, in the long run, be so good a computer as the child to whom these acts have meaning. An arithmetic program which enthrones drill and thereby engages to secure "100% accuracy" not only offers a threadbare course but also defaults in the one thing it promises.

I. CONCEPTS

A program of informational arithmetic will naturally include as one of its objectives the development of useful number ideas. It will seriously question the value of the number fact "7 and 5 are 12" unless all three of the numbers involved in the fact, as well as the fact itself, have real meaning. Accordingly, a program of primary arithmetic may well devote far more than passing attention to the learning of small numbers.

This is no slight achievement. It is easy to assume that when a child can say 7, write 7, and read 7, he offers sufficient evidence that he knows 7. This is by no means true.

In the first place, the use of the verb "know" in this case is misleading. A child does not at some moment of revelation suddenly come into full possession of the meaning of 7. His knowledge of 7 will grow with his growth and it is doubtful if this knowledge will ever be perfect. Moreover, there are many ways in which 7 may be known. Each of these ways is important, but no one of them necessarily implies the other.

For example, 7 may be known merely as an item in a series. As such, it lies between two other items called 6 and 8. Such an idea of 7 may be quite without number meaning. A child may know 7 in this sense without knowing that it is one more than 6 or one less than 8. He may not know that 7 is more than 4. At a certain stage he counts by rote. On that basis alone, the number names which he pronounces have no quantitative meaning. Why, then, should 7 be apprehended as greater than 4 merely because it is said later? The alphabet is a similar series. It never occurs to us to expect the child to think *g* is greater than *d*.

Seven may also be known as the sum of two smaller numbers. This is an important advance, its importance being based upon the assumption of a relatively vivid concept of each of the two smaller numbers. Seven may come to have still greater meaning when thought of for other purposes, for example, when thought of as composed of more than two small numbers, such as 3 and 2 and 2.

The idea of 7 is further enriched as it is apprehended through operations other than addition—as 1 less than 8, or 3 less than 10, as two 3's and a 1, as two 4's less 1, and so on. Somewhere in the development of the concept of 7 arises the idea that 7 is seven 1's. The full possession of this notion leads to the ratio concept, the wide application of which will engage attention over a long period of time. Each of these ways of thinking of 7 has its peculiar uses. Each contributes to the child's versatility.

The foregoing statements about the concept 7 have sole reference to the mathematics of the question—the way in which 7 is known as a number. Another element in the conception is the idea of universality. It is not difficult to believe that we are here at the very heart of the matter. Seven is something that does not inhere in the blocks or splints or buttons. It is much more widely applicable. Moreover, it is not something which the mind extracts from objects, but rather something which the mind puts into objects as it works with them. It does not exist in nature. It is man-made. In the case of each individual child it has to be made by him and for him. It is not made when we present him with the symbol 7. It is not made when we throw down 7 beads before him. It is made through his own activity. It arises as he is occupied in the manipulation of things for purposes which may have nothing to do with learning number. Here the doctrine of progressive education is profoundly right. The child must do his own work and it must be his work. No one can do it for him. The stage may be set by the teacher and, indeed, it must be if ground is to be covered. But the activity must receive the inner acceptance of the pupil.

The ways in which 7 may be known are capable of being considered functionally. Here the question is, What does the knowledge of 7 permit one to do? Broadly speaking, there are only two ways in which 7 may operate functionally—through reproduction and through identification. Either the idea of 7 is presented to consciousness and action takes place in accordance with the idea (reproduction), or else the situation is presented and the idea of 7 emerges (identification). This emergent idea may or may not be named. In either case it may provide the stimulus for a new activity of a reproductive nature.

Laying down a quantity of pennies, we say to the child, "Give me seven." If he does this successfully he reproduces 7. Similarly, if a girl when setting the table knows she needs to provide for seven persons she is reproducing 7 when she lays that number of places. If you tell the saleslady you want seven yards of gingham, she reproduces 7 when she measures it. It is not even necessary that the presentation should involve a number symbol. It may consist of a group of things and the reproduction may then be a matching process.[1] So in life children, youths, and adults go about reproducing 7, or 10, or 27, or one dozen, or ¾. They do all this in response to the number idea externally or internally presented. Their behavior, if it is correct, may then be said to be ***number-wise***.

On the other hand, we identify numbers. The teacher lays down seven pennies and asks the pupil, "How many pennies are here?" The correct answer is an identification of 7. Similarly the child (and the adult too) establishes in all sorts of activities the fact that here are 7, or 100, or 6¼, or 3.1416. Generally speaking, as soon as the identification is made some reaction appropriate to the number in question takes place. This reaction may be either external or internal, a physical or mental adjustment. It may be emotional

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[1] Matching which is carried out purely on the basis of one-to-one correspondence is not number reproduction. The child may set the table by laying one place for father, one place for mother, one place for sister Jane, and so on until the seven members of the family are provided for. In this operation, however, the number 7 is not reproduced because the idea of it is never entertained.

—a sense of familiarity or strangeness, of wonder or indifference, of exultation or remorse. The identification of number may be exact or it may be a mere estimate. The recognition of the approximate number of items presented for consideration may be all that intelligent action requires.

Informational arithmetic requires with respect to concepts that much more of the child's own activity be invoked than is expected where computational arithmetic holds sway. It likewise requires a much more careful and prolonged attention to these concepts—to the reproduction and identification of small numbers in a great variety of ways. One of the most ingenious ways of handling the transition between the use of objects and the use of number symbols is provided by number patterns. These may be dot patterns on cards. Because the dots may stand for anything—houses, people, dolls, jackknives—they are abstract. Because they may be seen as separate units they are concrete. This semi-concrete material has received considerable attention in the literature of arithmetic. Lay,[2] Freeman,[3] Howell,[4] and Brownell[5] come at once to mind, and there are a vast number of other investigators, mostly German, who have devoted attention to number patterns. Naturally much ingenuity has been expended upon the minutiae. Brownell's treatment, however, will be found to be especially helpful.

II. NUMBER FACTS

The number facts are themselves informational. Indeed, they are the informational basis of computation. All examples may be resolved into them. They are the alphabet of the number operations.

An arithmetic program which is dominated by the idea of computation is likely to offer the number facts as so many abstractions to be learned. The favorite method is to pound away at these by sheer repetition. Whether the pupil afterward invests these abstractions with meaning depends pretty largely upon the kind of pupil he is. Such a program leaves this to the individual as if it were relatively unimportant.

[2] Lay, W. A., *Führer durch den Rechenunterricht*. Leipzig, 1907.

[3] Freeman, F. N., "Grouped Objects as a Concrete Basis for the Number Idea." *Elementary School Teacher*, 12:306-14, 1911.

[4] Howell, Henry Budd, *A Foundational Study in the Pedagogy of Arithmetic*. The Macmillan Company, 1914.

[5] Brownell, W. A., "The Development of Children's Number Ideas in the Primary Grades." *Supplementary Educational Monograph*, No. 35. University of Chicago, 1928.

Informational arithmetic, however, by building up rich number concepts over a considerable period and stressing the relations of these numbers among themselves, succeeds in establishing the number facts in concrete form before they are elevated into consciousness as abstract facts. It is well known that in this manner the child comes to possess many number facts before he enters school. How did he learn them? Not by repeating "five and two are seven," but by vividly and actively knowing the meaning of 5 and 2 and 7. Confronted by a concrete problem calling for the adding of 5 and 2, he performs the addition as a mere expression of his competence with respect to the numbers involved.

The school can draw a lesson from the way the child learns outside the classroom. This does not mean that no numbers should be taught in the first grade. It means quite the reverse. The astonishing amount of number knowledge which children possess when they enter the first grade indicates the real situation. The debate as to whether number is too hard for first grade children or too uninteresting or too unimportant is futile. There was a point to the debate when the supposition was that children should begin with the number symbols and drill upon the number facts. But when we have in mind the developing of number ideas and relationships through motivated experience, the question of first grade number is not debatable.

Doubtless if rich number experiences could be carried far enough it would be unnecessary to resort to formal teaching and learning of number facts. Unfortunately, however, there is a definite job here to be done. The best activity program is likely to be relatively unsystematic. There comes a time when it is necessary to pick up the loose ends. Let there be no misunderstanding as to the objective at this point. The number facts must be learned.

When we take up the direct teaching of those number facts that need to be taught as number facts, let us proceed meaningfully. Let us relate the new facts to those already known—5 and 6 to 5 and 5 (assuming the latter to be already known), 9 and 7 to 10 and 6 (known), 12 less 7 to 12 less 6 (known); all subtraction facts to their corresponding addition facts; and all higher-decade facts to the parent basic facts.

Let us also adopt forms of expression which have meaning to children rather than forms handed down from the Middle Ages. For example, in multiplying, instead of adopting the phrase "seven times five" we may use the similar and more easily understood phrase, "seven fives"; in division instead of saying "thirty-five divided by five are seven," we may well prefer to say, "the number of fives in thirty-five is seven," or more briefly, "fives in thirty-five are seven," or (with a difference in meaning) "one-fifth of thirty-five is seven." In multiplication and division we also have an opportunity to extend that ratio idea of number to which reference has already been made. If 7 is "seven times whatever 1 is," it may be observed that 1 is now a group of 5's such as □ □ □ □ □, and that 35 is seven of these new 1's or seven 5's such as:

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In the teaching of number facts let us not neglect to use verbal problems. Many children on entering school can give the right answer to such a problem as "If you have five cents and your father gives you three cents more, how many cents will you have then?" Yet these children will generally be unable to answer the abstract question, "Five and three are how many?" The point is that the problem has more reality. With the material which it names and the scene which it evokes, the pupil is familiar. He identifies himself with it. He is the hero of the story, the fortunate possessor of five cents and the still more fortunate recipient of three cents more. The resulting inventory of his wealth is natural and inevitable. It is also correct. Good problems are an important aid in making number facts significant.

The doctrine, then, which the spirit of informational arithmetic enforces with respect to the number facts is that, in the first instance, they be developed informally through the development of number ideas and that, in the second instance, they be taught systematically so far as such teaching is required and that when so taught they be presented with the maximum of meaning. This meaning will be heightened by relating unknown to known facts, by employing significant forms of expression, and by using many interesting and lifelike verbal problems.

III. The Number System

Informational arithmetic takes the decimal system of notation seriously. It develops the idea of higher units, each being ten times its predecessor. The nature of the decimal system determines the way we express the number facts and the rules we apply in the operations. In reality when we say that eight and seven are fifteen, we are merely regrouping the objects into the two given groups in accordance with the particular number system with which we are concerned. In other words, the two groups of 8 things and 7 things do not serve our purpose. That purpose is better served by the two groups 10 and 5 which we then write as 15. By putting the 1 in the second position and the 5 in the first position we agree that this shall mean one 10 and five 1's. If we used an octonary system we should seldom need to recombine 8 and 7. We should write the result as 17, where 1 written in the second position would mean one 8 and 7 written in the first position would mean seven 1's.

It is a real difficulty in teaching children the rationale of the number system that we have only one scale of notation to present, namely, the decimal scale. Indeed, we do not present that scale in any form except the one now in vogue—a form which even among decimal systems is a recent and highly artificial development. The real fact is that the notation now accepted by the modern world, while it is a marvel of simplicity, has attained this simplicity as a hard-won achievement of persistent human thinking. The doctrines of place value, with the use of zero as a place holder and the so-called Hindu-Arabic numerals, did not win complete acceptance in the western world until the sixteenth century.

No one can contemplate the meaning and scope of the number system which we daily employ without realizing that the understanding of it by the oncoming generation demands something more than the turning aside for a day or two from the serious business of drilling on facts and processes. It demands something more in our textbooks than a few paragraphs—generally regarded as an intrusion upon the real business in hand—entitled "The Reading and Writing of Numbers." A just estimate of the importance of the numerical language in which all our quantitative ideas now find expression will associate that language with every phase of the learning of arithmetic. The forms of computation, for example, which we teach to school children are not the forms which were employed before the Hindu-Arabic system came into general use. Indeed, after that system was well known, many forms of computation were employed and struggled for survival. The rules of the game depend upon the structure of our number system. This relationship should be constantly in evidence. If it were constantly in evidence, less drill would be needed and more meaning would be achieved in our teaching of arithmetic.

The fact that we teach only one number system is a handicap just as it is a handicap in literary expression to employ only one language. Whether or not the theory of place value would be better understood by pupils if we gave them a glimpse of it as it applies to theoretical number systems, such as the binary, the quinary, the octonary, and the duodecimal, is a moot question. Perhaps in introducing systems which correspond to no human usage we should lose more than we should gain. We can, however, use some of our tables of measurement with good effect. The table of circular measure is really a sexagesimal number system—that is, a number system in which 60 is the base, just as 10 is the base of the decimal system. Suppose we write 5° 6′ 8″, the 5 in the third place means $5 \times 60 \times 60$ seconds. The 6 in the second place means 6×60 seconds. The 8 in the first place means eight 1's, i.e., 8 of whatever 1 is. In this case, of course, 1 is a second.

Again, our table of things—great gross, gross, dozen, and units—is a duodecimal system or a system to the base 12. When we write 5 G. 6 gr. 8 doz. 4, the 5 means 5×12^3; the 6 means 6×12^2; the 8 means 8×12; and the 4 means four 1's or four things. Notice that in writing the quantity 5° 6′ 8″ or the quantity 5 G. 6 gr. 8 doz. 4, we name each place. We do not do this in our decimal system of numbers. It is part of the marvelous economy of the decimal system that the place occupied by a digit indicates its name and value. Nevertheless when we *say* decimal numbers we name the places as we do when we say denominate numbers. For example, when we read 5684, we say the word "thousand" in connection with the 5, we say the word "hundred" in connection with the 6, and we say a broken-down "ten"(ty) in connection with the 8.[6]

[6] When a number is used for identification—as in telephone numbers—instead of

In pointing out the analogy between tables of measurement and our number system we do not need to limit ourselves to tables in which a constant factor appears—as 60 does in the table of circular measure. Consider the notation "4 yd. 2 ft. 7 in." This, of course, means $(4 \times 3 \times 12 + 2 \times 12 + 7)$ inches. Here we are dealing with groups of groups just as we do in our number system. Instead of the group at the left being tens of tens of units, it is threes of twelves of inches. It has been pointed out that even in geography we have a vague scale of notation—hamlet, village, town, county, state, country, continent.

These rather obvious statements are made here not because they express anything which may not have occurred to the reader before, but as an instance of the way in which matters which are usually kept apart may be brought together because of their basic relationships. If this is done, not only are all the parts of arithmetic welded together in support of each other but the whole subject takes on new meaning and dignity.

The number system, like the number concepts which it so admirably expresses, is entirely man-made. It is a hard-won achievement of the race. To thrust it upon pupils and demand that they use it in calculation before they know what it means is to place ourselves in an awkward position. In no other type of school work do we countenance mere verbalism; yet here we feel obliged to do so. We cease to expect thinking and are willing to accept ready-made formulas.

We say to the pupil, "Put down the 3 and carry the 2 to the next column," or "Add 1 to the figure in the next order of the subtrahend," or "Write the first figure of the partial product under the digit by which you are multiplying." In long division we say, "Notice that the steps are always 'divide, multiply, subtract, bring down.' What are the steps in long division, Henry? Jane? Clara? Class? Now you all know how to do long division." These summaries of procedure are admirable if they are understood. But if our object is to produce a thinking citizenry in a free community, then this blind imposition is an insult to intelligence. It is pathetic too that a child should be led to believe that because he is mechanically able to solve certain problems he therefore understands them.

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for expressing quantity we leave out the names of the orders. There is also a tendency to do this in other connections, e.g., one forty-seven, instead of one hundred forty-seven.

The practical teacher will ask, "Do you mean that the rules of long division should be explained?" This question is asked in the belief that no one will dare maintain that such rules can be understood to any useful degree by any but the brightest children. If the question is propounded with sole reference, let us say, to long division in Grade 4, then anyone who knows fourth grade children and school conditions will admit that understanding long division is impossible for most children. But if this is the case, one ought to inquire whether or not such a condition is inevitable. In the first place we should consider the grade-placement of long division. Washburne[7] gives evidence for putting it in the sixth grade, and recent trends in making courses of study seem to indicate that it will probably go over into the fifth grade.

Again, consider the background and insight which might be built up for a child by emphasizing understanding from the very beginning of his course in number. The point here is that the question whether to explain or not to explain should not first arise in a complex matter such as long division. It should arise as a question of policy with respect to the entire course in arithmetic.

It is not my intention here to contend that all pupils can understand the rules of computation. A protest, however, is entered against the defeatist policy of assuming that practically no children can really understand number and that therefore the only way to proceed is to state each rule and to drill in their application.

This defeatist attitude is back of the movement to postpone arithmetic until late in the school course. Some would begin it as a subject in the third grade, others would defer it still longer. Two observations may be made in this connection. First, the child has need for number and number operations long before he reaches the age to which these plans would postpone the consideration of arithmetic. Numerous investigations attest the fact that without instruction he acquires and uses number for his own purposes.[8] The doctrine, accepted without question in other subjects, that in-

[7] Washburne, Carleton W., "The Grade Placement of Arithmetic Topics." *Twenty-ninth Yearbook, National Society for the Study of Education*, 1930, pp. 641-670.

[8] Polkinghorn, Ada, "The Concepts of Fractions of Children in the Primary School." Master's thesis, University of Chicago, 1929.

Buckingham, B. R. and MacLatchy, Josephine, "The Number Abilities of Children When They Enter Grade One." *Twenty-Ninth Yearbook, National Society for the Study of Education*, 1930, pp. 473-524.

Connor, W. L.; see Buckingham and MacLatchy, *loc. cit.*, pp. 509 ff.

Woody, Clifford, unpublished manuscript.

struction should correspond to need and readiness, might be expected to apply in arithmetic. Second, if arithmetic as taught demands abilities which children do not possess and if children nevertheless manifest an early need for arithmetic in carrying out their own purposes, then it is more reasonable to change our program than it is to postpone it.

IV. LARGE NUMBERS

A course of study in arithmetic which confines its attention to computation is likely to belittle the importance of large numbers. The daily press is full of them. The huge operations of the government in connection with the N. R. A. and similar measures are front-page stuff. So also are, from time to time, values of building operations, losses sustained by fire, customs, receipts, foreign debts, populations, production figures of all sorts, unemployment, car loadings, and the like. The course of study which does not familiarize pupils with large numbers fails to meet their needs. It is not claimed that children can acquire concepts of these large numbers which will be so vivid and accurate as their concepts of small numbers. This is humanly impossible. But we do owe it to our pupils, as to ourselves, to take some pains to provide for them an approximate idea of such a number as a million. Galton did this in a notable passage in *Hereditary Genius*. He says:

> Permit me to add a word upon the meaning of a million, being an number so enormous as to be difficult to conceive. It is well to have a standard by which to realize it. Mine will be understood by many Londoners; it is as follows: One summer day I passed the afternoon in Bushey Park to see the magnificent spectacle of its avenue of horse-chestnut trees, a mile long, in full flower. As the hours passed by, it occurred to me to try to count the number of spikes of flowers facing the drive on one side of the long avenue—I mean all the spikes that were visible in full sunshine on one side of the road. Accordingly, I fixed upon a tree of average bulk and flower, and drew imaginary lines, first halving the tree, then quartering, and so on, until I arrived at a subdivision that was not too large to allow of my counting the spikes of flowers it included. I did this with three different trees, and arrived at pretty much the same result: as well as I recollect, the three estimates were as nine, ten, and eleven. Then I counted the trees in the avenue, and, multiplying all together, I found the spikes to be just about 100,000 in number. Ever since then, whenever a million is mentioned, I recall the long perspective of the avenue of Bushey Park,

with its stately chestnuts clothed from top to bottom with spikes of flowers, bright in the sunshine, and I imagine a similar continuous floral band, of ten miles in length.

This whole question of the presentation of large numbers in our arithmetic course is related to the problem of reading and thinking numerically. As has already been suggested not a little of the reading matter with which people are persistently confronted involves large numbers. To get the meaning of the passages in which these numbers occur it is not only necessary to have some idea of the meaning of each number but also to be able in some degree to compute with these numbers. If we read that the municipal budget for next year is $12,587,000 and that the budget for this year has been $14,440,000, it is not enough for us to apprehend even with considerable clarity the meaning of twelve million and of fourteen million. We must be able to represent to ourselves to a satisfactory degree the difference between these two budgets. If we are taxpayers, a sufficient approximation may be two million dollars. If we are officials in the city government, we shall want a more exact representation of the difference. Again, we are likely to want a more or less accurate understanding of the relative decrease of next year's budget from the budget of the present year. The general reader will note that the decrease is about 1/7, or 14%. Of course, the reporter may furnish the very figures which the reader wishes, that is, the difference between the two budgets and the per cent of decrease. On the other hand, he may not. Many writers, especially newspaper writers, have neither the time nor the inclination to embellish their statements with the results of computation. Even if they do this, the reader may well wish to verify roughly the computations—a procedure which sometimes discloses an error.

The common man needs to read quantitatively and to think quantitatively and, contrary to general opinion, his needs in this respect are by no means confined to small numbers. It is not enough that he should be able to compute with both large and small numbers. He should also be able to follow a line of reasoning, to detect fallacies, and to select pertinent data.

Joseph P. Day, writing on "More and Better Mortgages," in the *Saturday Evening Post* for February 6, 1932, showed that in spite of laws against usury, means are continually devised for charging exorbitant prices for the use of money. If, for example, a man wants a second mortgage, a certain organization will first charge him

28% in advance for costs. This means that if the face of the mortgage is $5,000 the borrower will get only $3,600. He then pays interest at 6% (the legal rate and the only one that appears in the transaction) on $5,000. Of course, he has to pay the entire $5,000 at the expiration date. The house owner may be able to compute—add, subtract, multiply, and divide—but can he think his way through a problem of this kind? Can he figure the real rate of interest to be paid in such a transaction? In this country millions of people, many of them unemployed, are paying vastly more than the legal rate of interest without knowing that they are doing so. This is notoriously true of those who buy on the installment plan.

With reference to the millions of people who are paying this high interest, E. R. Hedrick[9] says,

> The one thing they need to know is that compound interest works like geometric progression. Do they realize this? Doubtless every one of these millions had had geometric progression. Do they recognize it? Can they control the problems of their lives? Do you, for that matter, know the ruinous rates which are being paid by those women and men who have been told that their mathematical problems would be done for them by 'experts'? Well, they have been *done* by 'experts'.

V. FRACTIONS

Children come to school with some knowledge of fractions. Generally they have some notion of one half and many of them have a working knowledge for their own purposes of thirds and of fourths. The school, however, tends to begin work with fractions three or four years after the child has entered school. It then plunges directly into reduction and into the fundamental operations. The bulk of the course is on a computational basis. In reduction, you divide both terms of the fraction by the same number; in addition, you cross multiply; in multiplication, you multiply the numerators for the new numerator and the denominators for the new denominator; and in division, you invert the divisor and proceed as in multiplication. The children seldom know why these things are done, and in high quarters it is held to be useless to tell them.

Far more time should be spent on the concepts of fractional numbers. Moreover, this work should begin not long after similar work with whole numbers is begun. It should always be kept in mind that fractions are numbers just as truly as integers are. In fact, we have here the first of a long series of steps each of which enlarges the pupil's idea of number. In the first instance, number is merely the natural series of integers. Now the broken numbers or fractions enormously enlarge the number realm. The next great step is going to be taken when directed numbers are introduced. The incommensurables will then follow, then the imaginaries, then the transcendentals. Each of these steps enormously enlarges and generalizes the domain of number.

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[9] *Mathematics Teacher*, 25:259, May, 1932.

Like whole numbers, fractions should be considered functionally. The two great wings of functional treatment are reproduction and identification. Reproduction is action according to the idea of a number—in this case fractional. For instance, cutting the melon in half when one wishes to do so or when one is told to do so, is a case reproducing one half. Finding a stick 3¾ inches long, eating one-third of a chocolate bar, drawing a line one-fourth as long as a given line, filling a drinking glass three-fourths full of orange juice, tying two-thirds of the apples on the Christmas tree—these are cases of reproduction, that is, of carrying out one's own or another's idea expressed in fractional form.

Also, it is important that the pupil should recognize a situation as exhibiting a certain fractional number. In other words identification is needed. One sees a pie which has been cut and recognizes that one fourth of it is gone. The shorter of two pencils is seen to be two thirds as long as the longer. In reading a passage in geography one recognizes from the population figures that Milwaukee is one-fourth as large as Philadelphia. A picture of six doughnuts is shown to a child and he is asked, "If you had four of them what part would you have?" George measures Ann's height and reports it as 48¾ inches.

These and thousands of other activities are more or less available for school use. The purpose of them is to set the child to behaving number-wise with respect to fractions.

In all this it will be recognized that, just as in the case of whole numbers, so in the case of fractions a vast range of applications is possible. Part of the reason investigators find among adults so limited a use of fractions is that the application is so dimly seen. Among children the functional activities connected with fractions always imply material to work with, a medium through which the fraction idea arises or in which it is acted out.

Fractions should be used in connection with standard units of weight and measure, in connection with single objects which are divided or parted, in connection with groups of objects, and in connection with both concrete and abstract numbers. In short, fractions should receive a breadth of application to the end that children may regard them as easy and natural in all the media to which they belong.

No treatment of the concept of a fraction will be complete without referring to the new mathematics involved. Just as there are several ways in which a whole number may be known in a mathematical sense, so there are several ways of regarding a fraction. First, it may be regarded as one or more of the equal parts of a unit. According to this meaning, $\frac{2}{3}$ is two of the three equal parts of one. Second, it may be regarded as one (always one) of the equal parts of one or more units. According to this meaning, $\frac{2}{3}$ is *one* of the three equal parts of *two*, $\frac{3}{4}$ is *one* of the four equal parts of *three*, and so on. Third, a fraction may be thought of as division, its value being the result of the division. Here $\frac{2}{3}$ means two divided by three, and in expressing the value of $\frac{2}{3}$ decimally we actually carry out the division. Fourth, a fraction may be thought of as a ratio. As such it is not primarily a number but an expression of relationship. For example, $\frac{2}{3}$ expresses the relationship of two to three. Where this relationship holds, two numbers are, or may be, involved; and if they are involved, one of them will be two-thirds of the other.

How many of these meanings of a fraction should be introduced, and how early, depends upon circumstances. As matters stand today the first meaning receives a disproportionate amount of attention.

Throughout the entire elementary school the pupils' concepts of fractions should be slowly maturing. This growth should be accompanied by much activity and manipulation. Varied units and materials should be employed. Gradually the various mathematical meanings of a fraction should be introduced through constructive work. Pupils may be enabled to acquire the notion of the fraction *form* as a more general expression of number than the whole number itself, that is, that every whole number is capable of being expressed as any one of an indefinite number of fractions.

Again, the pupil may win the fruitful mathematical idea of the enlargement of the number realm through the introduction of fractions. How many fractions are there? Or in a more restricted sense, how many proper fractions are there? The idea of the infinite number of proper fractions in relation to the infinite number of integers (there is no last number); the realization that all these new numbers lie between two of the integers already known, namely, between zero and one; the conception of a similar infinite number of mixed numbers between one and two, between two and three, and so on without end—such notions are stimulating. Nor does there seem to be anything difficult about them. After all, arithmetic is a mathematical subject. Why should it not be taught and learned as mathematics?

VI. MEASUREMENT

How many days are there in a week? How many pounds are there in a ton? How does a sea mile compare with a statute mile? What is the difference between a dry quart and a liquid quart? What is the unit for measuring energy? What is the official tolerance for a foot rule? What constant is used in converting metric units of length into English units of length? These are matters of information. They cannot be derived by logic nor computed by numbers. They exist by agreement; one either knows them or does not.

We use information of this sort in countless ways, frequently without being conscious of doing so. Our knowledge that there are seven days in a week colors our thinking, modifies our behavior, and influences our plans. Moreover it enables us to accommodate ourselves to the plans, behavior, and thinking of others who have the same information. It guides us in judging events in the recent past and enables us to set them in order. It gives meaning to daily affairs through placing them in sequence with related occurrences which have already taken place or are soon to take place.

The act of measurement yields information or the data for information. A woman, interested in renting an apartment, measures the length and the width of the living room. The result is information which she can use according to her experience of rooms and her knowledge of her needs. A physician measures one's blood-pressure and gets information. A surveyor measures angular magnitude and gets information. A fisherman weighs his catch and gets information. By direct measurement we are continually answering the insistent modern question, "How much?"

In respect to measurement, informational arithmetic may thus be separated in thought from computational arithmetic. Yet it would be a childish mistake to suppose that such a separation exists in actual practice.

In the first place, computation contributes to information by affording practice. Our psychology—no matter what brand we favor—tells us that we know best what we use most. A child may be told that there are 231 cubic inches in a gallon, or he may read this fact in an arithmetic book. In either case the experience may be of little importance. What really matters is the use he subsequently makes of this equivalence, and among these uses probably none are so frequent as the ones which involve figuring. Thus this item of information is made his largely through computation. The scores of equivalences included, many of them in the form of tables, in our arithmetic courses, are items of information, but as personal possessions they vanish unless they are used. Much of this use is in the form of computation.

In the second place, computation may aid one in getting new information, that is, information new to the learner. The writer has lately been interested in studying certain features of the printed page, such as size of type and interlinear space. The printer's unit of measure for these matters is the point. The research literature, however, was not written by printers, and it does not use the point as a unit. It uses the millimeter. This literature cannot be used for practical purposes unless one knows the relation of a point to a millimeter. Finding no statement of the relationship, one has to compute it. Now a printer's point is $\frac{1}{72}$ (or 0.01389) of an inch. The familiar fact that a meter is equal to 39.37 inches furnishes the further fact that a millimeter is 0.03937 of an inch. Hence, by dividing, one finds that a millimeter is 2.83 points, correct to three significant figures. To the writer this was information, and computation based upon facts easily available had furnished it.

In the third place, not only does computation foster and even produce information, but information also serves the purpose of computation. The fact that the first four significant figures of the metric-to-English conversion factor for linear measure are 3-9-3-7 sets the whole world to multiplying or dividing. Meters are changed to inches by multiplying by 39.37; centimeters, by multiplying by 0.3937; and millimeters, by multiplying by 0.03937. If, however, the given data are in inches and metric equivalents are needed, we divide by the same significant figures, adjusting the decimal point according to the particular metric units we desire.

Thus the close relationship between informational and computational arithmetic is illustrated. We use the information in a great many ways, some of them subtle and unconscious. But one very definite use is for purposes of computation. This computation itself is a process involving certain facts, rules, and skills.

On the other hand, computation, as has been abundantly shown, also serves the purposes of information. While the inter-relationship between the two aspects of arithmetic has here been illustrated with reference to measurement, the relationship is likewise evident throughout the entire field of arithmetic.

There are, however, many aspects of measurement which are peculiarly informational, that is, informational in a sense in which the other topics of arithmetic are not. Indeed, it is not too much to say that in the direct and obvious sense of the term, information is provided in greater degree with reference to measurement than it is in any other aspect of the arithmetic course. Much of this information we try to impart to children as a permanent possession. There are 12 inches in a foot. This item of knowledge is so commonly required that one would be at a disadvantage without having it. On the other hand, many of the facts of measurement are properly to be regarded as reference material. It is desirable that much of this material should be presented and used in the arithmetic course for the broadening effect which may thus be secured. The bulk of our population will never use any units of money except dollars and cents. Yet it is a threadbare sort of course which would give no acquaintance with the English, French, and German units, to say nothing of rubles, lire, and gulden.

On the side of information, then, arithmetic may be expected to use a great many facts of measurement for broadening purposes without carrying the treatment of these facts to the point where their permanent personal possession may be expected.

Space in textbooks—until fashions change and larger books are demanded—can hardly be devoted to a treatment of the many socially valuable items of measurement information. This does not, however, mean that the course as administered need omit them. Individual and group projects, for example, may deal with shopping in Paris, London, or Berlin. The appropriate money can be employed. Prices may easily be secured from foreign catalogues or newspapers. Judgment as to the value of the goods offered or purchased will require approximate conversion into United States money—say a shilling at 25 cents, a franc at 6½ cents or 15 to the dollar, a mark at 30 cents.[10] A rudimentary notion of exchange and its fluctuations, and perhaps of import duties, naturally arises in this connection.

Few children will ever use units of weight other than those of the avoirdupois system. Yet the course should not neglect, for informational purposes, either the apothecaries' system or the Troy system. A committee of the class may report on each of these matters, not forgetting to interview druggists and jewelers.

Similarly, the cultural value of finding out how the druggist measures fluids, how the sailor measures, how the lumberman measures, how the printer measures, how the paper industry measures ought to be recognized. Tables should not be memorized, nor should computations be made, except as they help in arriving at the cultural values.

Children will be interested in certain measures which are very commonly employed but whose units are seldom understood, e.g., the units used in measuring hats, collars, gloves, and shoes, to mention only articles of clothing.

The importance, from an informational point of view, of the metric system needs no emphasis. The fact that it is the legal system in practically all European countries indicates that it cannot be neglected.

The sense of long development is one of the evidences of culture. Measurement has a history and children should not be denied access to it. One of the ideas which they will find interesting in this connection is the way early man used parts of his body as units. They will find, upon investigation, that some of these units with or without change of name have been standardized, e.g., the foot and the ell (yard); while others, as the pace, the hand, and the finger, are still used very much as they were in primitive times. Some children may be interested, perhaps at the time they study "European Backgrounds of American History," to find out how people of other times measured and what sort of money they used.

Certain equivalences have grown up in the processes of trade and commerce which may properly engage the attention of pupils. They need not be memorized but their existence and general meaning should be known. Moreover, pupils should know where they may find, when they need them, these and other measurement facts—such equivalences as 231 cubic inches in a gallon, 62½ pounds in a cubic foot of water, 196 pounds in a barrel of flour, 60 pounds in a bushel of wheat, 500 pounds in a bale of cotton, 7½ gallons to a cubic foot of water, and so on. How far material of this kind is used depends upon the interests of the children. So far, however, as such facts are required in solving problems they should, in general, be given as part of the data.

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[10] These equivalents are the rough ones used by tourists in the summer of 1934. They served at that time every practical purpose. The German equivalent assumes the use of the "registered" mark, otherwise the equivalent would be about 40 cents.

It is evident that in the field of measurement informational arithmetic has much to offer. Indeed, it would be possible to write this paper entirely with reference to that aspect of information. One would begin by showing how small children in the primary grades can and should measure not only to serve the purposes of their activities but also to serve the purposes of computation. For example, the teacher who neglects the foot-rule as a means of vitalizing the notion that 2 and 3 are 5, or the fraction ideas ½, ¼, ¾, and so on, is leaving out of account a powerful instrument of instruction.

The treatment of measurement in our arithmetic courses falls far short of accomplishing the broadening effect which might be secured. In textbooks the treatment is usually uninteresting. Such a quantity of exercises in computation must be included that there is no room for the fascinating story of man's adventures in measuring. Moreover, the entire course from the third to the eighth grade must, as a rule, be offered in three books, each of which can be sold for less than a dollar.

The result is that only the barest and most rudimentary treatment is offered. Within the scope which the course usually contemplates, the presentation is sketchy and even erroneous. For example, there is no distinction between a dry quart and a liquid quart, although the former is about 9½ cubic inches larger than the latter. No adequate distinction is made between the pound troy and the pound avoirdupois. The fact that the one has 12 and the other 16 ounces does not dispose of the matter since their ounces are not the same (ounce troy is 480 grains; ounce avoirdupois is 437.5 grains). The gallon is often represented in problems as a measure of berries, yet it is not a unit of dry measure and when used as such instead of the half-peck it is in error by nearly 38 cubic inches. No inkling is given in our textbooks of the system of tolerances in weights and measures. Children are allowed to suppose that a foot is exactly a foot, a gallon exactly a gallon, and a pound exactly a pound. This is never the case. Accordingly, under the leadership of the United States Bureau of Standards limits have been set up. A foot must not vary from the standard foot (or the approved copy of it in the hands of the sealer of weights and measures) by more than $\frac{1}{32}$ of an inch, a gallon must not be "short" more than a dram (0.9 cubic inches), nor a pound by more than $\frac{1}{8}$ of an ounce.

But it is in the more humanistic aspects of measurement that the arithmetic course is most deficient. Why should linear measure be introduced by a table and disposed of by abstract examples in conversion? Why not tell the story of the efforts which man made to measure length and the meaning of the units which have come down to us? The story cannot be told here. One may suggest, however, the curious rôle of the barley corn and why it was so generally adopted in various parts of the ancient and medieval worlds as the first or smallest unit of length—how, too, it is the origin of the grain as a unit of weight. Then the persistent use of parts of the body—any parts which definitely terminate in ends, sides, or angles—ought to be referred to. These were for things to be handled. For distances to be traveled, the body in action was naturally employed. Hence arose the pace (single and double) and the day's journey. All these units were established independently of each other. The finger, for example, was really and truly the breadth of the forefinger. The palm was the actual breadth of the hand without the thumb. The span was the distance from the tip of the little finger to the end of the thumb when the fingers were outstretched. The cubit was the distance from the elbow to the end of the middle finger. Later some systematizer put things like these together in a table. The Hebrews, for example, had these equivalences: 6 barley corns = 1 finger, 4 fingers = 1 palm, 3 palms = 1 span, 2 spans = 1 cubit. Likewise two cubits were an ell, which, however, had an independent existence as the distance from the tip of the nose (or the middle of the breast) to the end of the middle finger when the arm and hand were extended. The ell is essentially our yard, although the English yard is more properly derived from the Anglo-Saxon girth.

It is clear that informational arithmetic can offer much in developing the story of measuring length. It can do so equally well in connection with measures of capacity, of weight, and of time. Nor is the story one of history alone. The uses of measurement in modern society are likewise splendid material for informational arithmetic.

Henry D. Hubbard[11] says

> Measuring tools evolve from simple origins—a stick, a shadow, a gourd, a stone—to-day they number thousands. They give us new senses, enable us to detect invisible light, to feel magnetic forces, to sense a thousand things otherwise imperceptible. With the crescograph one can see the movements of growing plants. A lens-and-mirror device measures the diameter of stars, a feat comparable with measuring a coin fifty miles away. The radioactivity of radium is measured by the motion of an electrified strip of gold foil, while electrified quartz threads are sent up ten miles to measure cosmic rays.

Science also makes a vast number of what may be called natural measurements—

> . . . melting points of solids, weights of atoms, lengths of light waves, properties of materials. In such measures lie latent the means to alter nature almost at will and make it serve our purpose with almost magical power. Such measures build civilization. They are the numbers which rule the world of enterprise, the unseen frame of all achievement.[12]

"Mathematics is queen of the sciences and arithmetic the queen of mathematics." This famous remark of the master mathematician Gauss referred to a type of arithmetic which the schools, no matter how advanced, seldom teach. Among the Greeks arithmetic was highly valued as a study but it was not the arithmetic of reckoning. That was logistic and little regarded. In the Middle Ages arithmetic was one of the Seven Liberal Arts, but it was a theory, not an art.

It is evident that arithmetic has meant many things to many men. To Gauss it was a great mathematical subject. Can we not so far accept his dictum as to attempt to lay, however humbly, a really mathematical foundation in our elementary school? To the Greeks and to the philosophers of the Middle Ages arithmetic was a discipline. Can we not adopt their position also, to the extent of asking that our arithmetic shall make its contribution to a liberal education? Besides all this, can we not give our own interpretation to arithmetic and make it the means by which, even from the first school days, the child acquires information about number and an ability to think in quantitative terms?

[11] Hubbard, Henry D., "The Romance of Measurement." *Scientific Monthly*, 33: p. 357. [12] *Loc. cit.*, p. 357.

THE RELATION OF SOCIAL ARITHMETIC TO COMPUTATIONAL ARITHMETIC

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I. An Overview of Arithmetic

An overview of the development of any field of knowledge reveals certain major trends which, when studied critically, tend to simplify one's thinking about current problems relating to that field. Such an overview of the subject of arithmetic is particularly illuminating.

Trends in the past. In the schools of the early colonial period arithmetic received little attention. The subject made its appearance gradually, due to practical demands, until by the year 1800 arithmetic was a common subject in elementary schools. However, as revealed by the textbooks of the period, the subject was taught in a very formal manner, the instruction consisting primarily of the memorization of rules followed by their application. There was no attempt to develop a rational understanding on the part of pupils. In 1821 Colburn published a set of arithmetics which instituted radical reforms. The teaching by rules was abandoned, oral instruction was emphasized, some drill was introduced to secure better mastery of the number combinations, and a rational understanding of practical problems was attempted. The result of these reforms was a great expansion of the subject of arithmetic. In the main the nature of this expansion was desirable up to approximately 1860, following which time, while the subject continued to expand in terms of the proportion of the school day used for arithmetic, the nature of the instruction became increasingly formal until by 1890 progressive educators were again demanding reforms.

From 1893 to 1910 there were general demands for the reduction of the amount of time given to arithmetic and for the elimination of much socially useless material. The criterion of social utility was applied, those topics which were not considered of social value being eliminated. One should note particularly, however, that this term "social utility" was interpreted purely in terms of computational values. While there have been serious checks to the application of the theory of social utility to the subject of arithmetic since 1910, it is still common to consider a topic as socially valuable only when there is evidence of rather widespread computational use.

One other major element revealed by an overview of arithmetic is the very large amount of emphasis given to drill through the use of formal practice exercises, beginning with the early practice exercises of Courtis and Studebaker and extending well up to the present time. This enormous use of drills reveals again the large degree of importance attached to computational ability. While all along there have been occasional apostles of a type of arithmetic which would emphasize non-computational values, it is only within the last half dozen years or so that a redefinition of social utility has received any widespread attention.

In 1926 McMurry gave concrete expression to his reaction against the prevailing character of arithmetic by publishing a series of books in which a deliberate attempt was made to secure a much greater emphasis upon social values. While this particular set of books has probably been more effective in influencing the thinking of students of arithmetic than in changing the character of practice in public schools, it at least attempted to break away from an over-emphasis upon computation. In several of his writings Judd has repeatedly attacked the formality of arithmetic and has emphasized the values which might come from a genuine socialization of this subject, stressing particularly the use of arithmetic as an instrument of thinking rather than as simply a tool to facilitate computation. Again, in the *Twenty-ninth Yearbook of the Society for the Study of Education* the reviewing committee under the chairmanship of Brueckner called attention to the need of greater emphasis upon non-computational arithmetic. Under the headings of "The Informational Function" and "The Sociological Function" of arithmetic, this committee made a plea for a type of treatment which would make a pronounced change in the nature of the social values to be derived from this subject.

The present situation. The present situation may be expressed briefly as follows. Twenty years of excessive use of drill and practice exercises have produced an emphasis upon rapid computation to the exclusion of any considerable amount of emphasis on quantitative thinking in social terms. The enthusiasm for practice exercises, which was evident in 1920, has waned rapidly during the last decade. The persons who have been operating on the course of study in arithmetic under the theory of social utility have based their estimates of social utility upon surveys of the computational uses of arithmetic and have written as though their private interpretations of social utility were the only acceptable definition of the term. The writer of this chapter challenges their definition of the term "social utility." As the term "social utility" has been commonly used in the subject of arithmetic, it refers to those computational processes which have been found commonly used in various kinds of business practices, as revealed through surveys of business usage. This narrow definition of "social utility" disregards entirely the fact that computational practices may be carried on with no understanding and in an entirely formal manner. It evaluates arithmetical ability in terms of an adding-machine. It fails completely to comprehend that genuine social values of arithmetic are related to the higher rational processes of understanding the significance of quantitative situations. It is as formal in its implications as was the arithmetic of 1800, before the reforms instituted by Warren Colburn. If the practices of the last twenty years are carried to their logical culmination, the result will be a type of arithmetic which will enable children to do in a limited number of seconds a certain number of computations which have been found to characterize the formal computational experiences of a limited number of groups in society. It will make it necessary for some other subject in the curriculum to assume the teaching of how to deal with those quantitative situations and relationships which characterize the thinking experience of all persons, but which do not eventuate in any particular computation done with pencil and paper, such as may be counted and measured in a survey of arithmetic practices.

The essence of the current problem can be stated thus: Computation is a necessary function of arithmetic which no one proposes to discard. Computation should grow out of as thorough an understanding of the number system and the operations involved in its use as may be obtained at any given level of maturity in the schools. It is not a problem of computation *versus* social arithmetic, but of the *relative* emphasis to be given the two. The writer believes that computation has been given much greater emphasis in the total amount of time devoted to arithmetic than it deserves, or than it has used effectively, and that the opportunities for improving arithmetic lie in the vitalizing of children's thinking in both non-computational and computational situations, but particularly in those non-computational situations which are far more common in experience than are those involving computation and which are also more important in determining the social behavior of human beings. It is the writer's belief that rapid computation will more and more be transferred to adding machines and calculators and that society can make an immensely profitable trade by devoting a considerable portion of the time which has previously been used for improving speed in computation to the development of a type of quantitative thinking which is essential to living in a modern world.

At the risk of presenting examples which are less effective than someone else might propose, the writer will attempt to illustrate in four specific situations the point of view which has been expressed in the previous paragraphs.

II. The Social Treatment of Arithmetic Topics in the Upper Grades

Selection of topics. In the textbooks for Grades 7 and 8 it has been customary to organize the arithmetical material presented around certain topics of probable social value, such as insurance, taxes, banking, investments, installment buying, etc. An application of a broad theory of social values to this portion of the course of study would result in two possible changes: first, some changes in the selection of topics considered to have social value for boys and girls with an age range of from twelve to fifteen years; second, a pronounced change in the character of the treatment accorded the topics which are selected.

In regard to changes in the list of topics presented in upper-grade arithmetic it must be conceded that some desirable changes have been gradually appearing during the last two decades. For example, the treatment of carpeting and paper hanging, which was customary in the textbooks of 1900, has to a very considerable extent disappeared altogether, due to the fact that the problem of carpeting floors has changed since that time and the fact that the old method of computing the amount of wall paper needed never did agree with the methods actually used in the wall-paper trade. The main criticisms of the topics which are now common are of two sorts:

first, the list of topics is altogether too limited, a number of excellent ones being left out altogether, and, second, while the topics used appear from their name to have social value the actual treatment accorded them in many cases gives no expression to the possible social values which might be obtained, but rather uses the topics simply as a vehicle for continued drill in computation.

In regard to the first criticism, the treatment of stocks and bonds as a form of investment undoubtedly has some possibilities of social value. However, the common treatment of these forms of investment is rather remote from the experience of average children, whereas an introduction to savings and investment by reference to the postal savings bank would be much more immediately related to their experiences and would possess a social value which is difficult to secure through the ordinary method of teaching stocks and bonds. Regardless of whatever obligations an author of a textbook may feel in regard to treating stocks and bonds, it is altogether probable that studies of price fluctuations and their relation to actual values possess all of the arithmetical possibilities involved in the business dealings of adults with the additional fact that such a series of topics might lie entirely within the range of experience of seventh and eighth grade children. The type of quantitative thinking which organizations such as the Consumer's Research group attempt to secure illustrates what arithmetic might contribute in the way of non-computational experience.

Treatment of topics. Even more important than a selection of topics for the seventh and eighth grades is the treatment accorded to the topics finally selected. Installment buying, for example, has been used frequently as a vehicle for obtaining further practice in arithmetical operations, but there are only a few books in which the types of quantitative judgment important for installment buyers are at all emphasized and in which the social thinking necessary to guide one's practices in this particular respect is provided. The practices of installment buying afford one of the best examples of the need for a type of quantitative thinking which goes quite beyond the ordinary computational problems which are pertinent to this topic. The treatment of taxation, as it commonly appears in the textbooks on arithmetic, makes a very small contribution to the type of judgment which a good citizen should make to a new taxing proposal. In the section of one textbook dealing with taxation the writer finds such problems as:

1. At a tax rate of 19.6 mills on the dollar, what is the tax on property assessed at $10,000?

2. Find the amount of the tax bill when the valuation is $10,000, the rate per $1,000 is $12.60, and the collector's fee is 1 per cent.

While these computations may represent the type of arithmetical activities in a taxing office, they do not represent the nature of the problem of taxation for the average citizen. When one reads in the morning paper that the tax rate for the ensuing year is to be cut by a flat reduction of 25%, a certain type of mental reaction follows. If the rate of reduction had been stated as 10% or as 45%, the mental reaction of many individuals would be no different, although the importance of the difference for their community might be very great. When one reads that the federal government has set aside $1,000,000,000 for certain purposes one's judgment should reflect some notion of the tax burden which would be represented by such an item as $1,000,000,000. In other words, the socially important problems relating to taxation are not those of a computational nature such as are figured in the tax assessor's office, but rather those problems involving quantitative thinking, for the most part of the non-computational kind, but which precedes the expression of whatever judgment the individual finally arrives at in regard to the point in question. A genuinely social treatment of the topics of the seventh and eighth grades should eventuate in a high type of quantitative thinking rather than simply afford further practice in computation, using the topic as a vehicle for such practice.

III. The Use of Problems

A second example of possible improvement in the social treatment of arithmetic relates to those problems designed primarily for maintenance of processes which have been taught previously.

Miscellaneous problems. As one examines a present-day arithmetic he finds that there are certain problems which are used in the explanation of new processes or operations and which must be selected for the particular needs of that stage of the explanation. It is quite to be expected that in such a case the problems may be unrelated to one another since the important matter to consider is their applicability to the particular point needing explanation. However, in those sections of the book in which several pages of problems are given for review there is a possibility of obtaining an increased social value by relating the problems to some major project which is socially worth while and which evokes a body of social information that is needed for clear thinking. Both types of treatment may be found in current textbooks. As an example of the miscellaneous organization of problems, the writer finds the following topics presented in successive problems on a single page of a textbook which he has before him:

Problem 1. Find how much a baby gained in weight during one month.

Problem 2. Find how much longer it takes one boy than another to run a given distance, the rate being stated.

Problem 3. At a given age for starting to school, determine how long it will be before a certain child who is now four years and five months of age will be able to begin school.

Problem 4. Determine how many potatoes are left after a certain number of bushels have been sold.

Problem 5. Find how much it will cost each member of a church to pay off a given amount of debt.

Problem 6. Determine how far a train will go in a certain number of hours at a given rate.

Problem 7. Find how many chickens a farmer must sell to pay for a given bill of goods which he has purchased.

Problem 8. Find how much larger one field is than another, the measurements of each being given.

Considered as isolated exercises for providing drill in computation these problems may be justified as they are. Considered in terms of the additional social value which might be derived from a page of problems involving exactly the same arithmetical operations, the problems may be criticized very pointedly.

Problems socially significant. In contrast with the method of treatment just described, reference may be made to another textbook in which on a single page one finds the following type of material. About a third of the page is given to concrete information showing how crowded certain business areas of a city may be as compared with certain residential areas. A diagram is presented showing one floor of a large office building, together with additional data giving the number of floors and the general dimensions of the building and the lot on which it stands. Comparable data are also given for a set of apartment house buildings and comparisons are drawn between the crowded conditions in these buildings. The pupil is asked to supply similar data for the school building which he attends. Following these basic items of information are seven problems which as far as the arithmetical computations are concerned might parallel the eight unrelated problems noted in the book previously mentioned. The significant fact is that after working the seven problems involved in the second book the pupil not only has his computational practice, but he has had brought very vividly to him the dense population of a skyscraper as compared with city apartment houses and with one school building, the one in which he happens to be. Having at hand these basic items of social information the pupil is better able to think clearly in regard to one of our very important social problems, namely, the distribution of our population.

If the type of treatment given on these two typical cases were multiplied by fifty, the amount of such work which might be found in a representative textbook, the net result would be, in the first book, a body of miscellaneous information which would lead nowhere, the only value realized being the arithmetical computation which is facilitated, whereas, in the second case, it would be fifty pages of experience in dealing with significant social data. The results of one or the other kind of training may result in the difference between a person who thinks on the basis of accurate information and the person who tries to think without the necessary quantitative information and whose conclusions are so frequently out of all accord with the facts.

Textbook makers are missing one of their genuine opportunities in failing to socialize the more or less miscellaneous type of problem which most of them include in their books. Some recent books exhibit a very commendable tendency in this direction.

IV. MEASUREMENTS AND DENOMINATE NUMBERS

A third opportunity for increasing the social value in arithmetic lies in the treatment of measurement and denominate numbers. The treatment of these topics may range all the way from a formal presentation of facts and tables, followed by their application in problems, to an interesting informational treatment regarding the nature of measures and denominate numbers, where they came from and why, together with a mastery of their quantitative relationships. Perhaps no better illustration could be given of a desirable form of treatment than to refer to some of the pamphlets published recently by the American Council on Education in which one finds an interesting body of information—the story of weights and measures, the story of our calendar, and a treatment of the development of devices for telling time. These pamphlets, ranging in size from thirty-two to sixty-four pages, present a wealth of material which any teacher in the intermediate and upper grades would find exceedingly useful in socializing the treatment of measurements and denominate numbers. As a social achievement of the race, the development of weights and measures and the development of mechanisms for telling time represent achievements of far greater importance than the average man realizes. The formal statement that sixty seconds equals one minute and that sixty minutes equals one hour is necessary information, but the teacher who teaches time by simply demanding computational ability involving these facts is missing the entire social significance of this topic in measurement and is at the same time depriving her pupils of a most interesting body of information. Teaching time relationships can be motivated greatly by the story of the invention of time-telling instruments, ranging from the earliest sundials, water clocks, sand clocks, candle clocks, etc., to the more intricate mechanical devices eventuating in the present types of clocks and chronometers. The one-hundred-yard dash in the stone age was never measured in split seconds, nor would it have been possible to start schools, had there been such, at precisely nine o'clock. The relationship between modern social living as related to scheduled time and the behavior of primitive groups is a fascinating story, possessing a significance for quantitative judgments of time far too great to be missed. Children may use a foot rule for measurements indefinitely without ever coming in touch with information as to how a foot happened to be as long as it is and when that unit of measure was first established. The establishment of any unit of measure is a story of social significance which the school has ordinarily overlooked, but the social importance of which is clearly recognized by the government in its Bureau of Standards.

V. Quantitative Information

The fourth and last suggestion for socializing the content of the arithmetic course is that a textbook in arithmetic might legitimately provide a certain body of quantitative information of significance in dealing with certain types of problems. The purpose of this provision would be to initiate the practice of basing one's thinking upon a body of quantitative experience. The application of this suggestion would result primarily in non-computational rather than computational activities. To be specific, one of the basic problems of society relates to the distribution of population. Children, of course, know that there are large cities and small towns, but a vivid understanding of the relative size of different places or districts can scarcely be obtained without some experience in comparing localities of different sizes.

As the writer is preparing this portion of his paper, a football game is in progress on a neighboring field. The morning paper stated that a crowd of 40,000 persons was expected. How large is a crowd of 40,000 persons? How does a child, living in a town of 5,000 who reads about the game and finds a statement that there were 40,000 persons in the grandstand, obtain any adequate quantitative concept of the size of the crowd? He may do a bit of computation and arrive at the notion that there were eight times as many people at the game as there are people living in his town, although it is doubtful whether such a statement would be very meaningful to him since he has never seen all of the 5,000 people in his town congregated in one place at one time. He knows from other sources that a certain city has a population of 1,000,000 and another city a population of 3,000,000, but these figures are more or less meaningless to him until he finally arrives at some comparable scheme in which these various sizes of population are arranged.

It seems that one of the functions of arithmetic should be to assist in the building up of more vivid concepts relating to quantitative thinking. A simple table of population is not sufficient to give vividness, but some analysis of the populations may do so. No other subject in the curriculum is attempting to build up for the child such bases for quantitative thinking. The difference in those who think quantitatively and those who do not is not expressed in terms of their relative computational ability but rather in terms of their habitual modes of thinking. The contribution of arithmetic to methods of quantitative thinking regarding social and economic problems is so significant that it should not be subordinated to an attempt to secure more rapid computation. The argument here is that a certain part of the space in an arithmetic textbook might justifiably be devoted to pure information of the quantitative sort which would be useful in the thinking experiences of children. Very little of this type of material is to be found in arithmetics now available.

VI. CONCLUSION

Arithmetic should be judged primarily in terms of its social values. These values are not to be determined alone by a survey of computational practices. The fallacy of the previous application of the theory of social utility lies primarily in the narrow and restricted definition given to social utility. The narrow limitations of this definition the writer entirely repudiates. Arithmetic as a method of thinking possesses fully as much social value as arithmetic as a tool for computation. The subject must be relieved of the application of such a narrow point of view. Arithmetic consists of two major types of material, first, a number system which must be learned as a system with all of its common interrelations as expressed in the operations of the four fundamental processes. Computational ability is essential and necessary for this type of mastery. Second, arithmetic consists in the socialization of this type of number experience until it permeates the common thinking practices of individuals. Overemphasis on computation has produced a lopsided arithmetic. The recent movement to balance the teaching of arithmetic by giving increasing emphasis to its social and informational values is a movement so significant that it may well become the outstanding reform which this generation will contribute to the subject.

OPPORTUNITIES FOR THE USE OF ARITHMETIC IN AN ACTIVITY PROGRAM

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Systematic and Functional Approaches to Arithmetic

The important rôle of arithmetic. The Committee believes that some facility in arithmetic is indispensable to the life of any normal child. Each week brings scores of situations in which a child must make use of number to carry on his work and play: he must be able to count his marbles or the number of children invited to his party; to measure the ingredients for a batch of candy, or the length, width, and thickness of the board for a model ship; to handle money in his purchases at the store, or to add up the cost of his lunch or total the cost of an order for stamps; to divide equally, or otherwise, the contents of a bag of fruit among his friends, or apportion the cost of an entertainment among the members of the group; to keep the moving hands of the clock in mind in order to terminate the music practice or to know when it is time to hasten home to eat; to differentiate between more and less, larger and smaller, heavier and lighter; etc. For these and countless other everyday activities in school and out, children use number to solve their difficulties and promote their interests. Arithmetic is, as we have said, indispensable. There is no difference of opinion on this point between those who would have children learn through an activity program and those specialists who hold that arithmetic should be taught systematically through a separate subject approach. And because arithmetic does serve so important a rôle, all educators desire that the child shall acquire his use of number by the most efficient learning techniques and shall be able to use these acquisitions at later times to enrich his experiences of a quantitative and qualitative nature.

Purposes of this paper. The purposes of this paper are to present:

1. A point of view held by a minority group in elementary school work concerning the learning of arithmetic.
2. A survey of opportunities for use of arithmetic in the "activities" curriculum.
3. Recommendations for the improvement of present practice.

There has been considerable innovation in curricular practice in elementary schools of America during the past decade. Innumerable public and private schools have attempted to break down the concept of formal education by bringing within the range of schooling the experiences and activities of normal child life. The typical child possesses a variety of interests and curiosities, a readiness which motivates him to participate actively in enterprises with almost tireless energy. Normal children find no particular "learning difficulty" in improving their roller-skating, or in mastering the rules and plays of a game of checkers, or in reading and following directions for assembling the parts of a model airplane. When the child has the purpose to achieve, those learnings which might otherwise involve difficult obstacles are with amazing agility mastered as necessary skills for achievement. And in those experiences where readiness attends the act, the learnings are freer to become a part of the next learning situation, to function when and where needed.

These curricular innovations and experimentations have led to many new departures in classroom practice. Education is now recognized as consisting of every activity in which children engage in and out of school. In many schools the daily program is not divided into separate periods for arithmetic, geography, spelling, composition, etc. In the classrooms of these schools education starts with an interest which absorbs the pupils and expands and deepens that interest into a series of related and worth-while experiences. For example, a group of children may be enthusiastic about aviation. They may read some of the ancient stories and myths about flying—"Daedalus and Icarus," "The Magic Carpet," or "Pegasus." They may follow Commander Byrd to the South Pole and eagerly watch through newspaper and radio his airplane adventures into the unknown Antarctic. Undoubtedly, the group will write for aviation bulletins and seek information from various sources. Some of the group may plunge into the theory of air dynamics or experiment with the principles of the rocket plane. Thus the children develop joyfully a series of meaningful and related aviation experiences which will differentiate eventually into a great array of useful skills, facts, understandings, and attitudes. Yet, the children may not be aware at any time that they are really pursuing what is in the formal curriculum classified as "courses" in literature, geography, composition and spelling, science, arithmetic, etc.

It is obvious, however, that such a series of aviation experiences would demand arithmetic at many points. For example, Byrd's trip has to be financed; his cargo has weight and takes space, his ship is small, yet he must provide food, shelter, clothing, and equipment for a definite number of men and animals for an extended period of time. The determination of these supplies is largely arithmetical. Any study of air dynamics or rocket planes will necessitate computation of a high order. Determining lift per unit wing surface, horsepower of the engine, fuel capacity and consumption, pay load, cruising range and speed, and a multitude of other problems can be solved only by the use of number.

A first viewpoint: functional experiences adequate. In schools where such related experiences constitute the curriculum, there is a difference of opinion in regard to the teaching of arithmetic. At one extreme are the teachers who believe that arithmetic should be taught only as it is needed to carry out the children's purposes in any given situation. These teachers contend that any live, forward-moving activity offers ample opportunities for learning the meaning and use of arithmetic. They would not make use of formal textbooks nor would they set aside a period in the daily schedule for drill in arithmetic. Should a problem of average speed arise in an aviation activity, the teacher and children would compute the answer and pause long enough to grasp the meaning of the arithmetic process involved in the solution. They declare they would not detract from the larger outcomes in the experience, however, by constructing a number of hypothetical problems in average speed and then practice the process in order to fix the particular arithmetic learnings used in the solution. This practice, they hold, would interfere with the larger drive—the aviation interest. This group of educators leaves the mastery of any number skill or process mainly to the dynamic quality of functional situations and argues that if educational experiences are sufficiently rich and varied, those skills and processes that are truly important are mastered by the demands of constantly recurring life situations.

A second viewpoint: systematic arithmetic mastery the goal. A second point of view concerning arithmetic and the activity curriculum starts from a different set of assumptions. In many classrooms, a definite body of arithmetic knowledge and skills is selected in advance to be learned. The teacher then seeks to find or to stimulate the children's interests which will demand the arithmetic content which has been selected for the year and to develop these interests into activities largely for the purpose of giving meaning and readiness to the arithmetic lesson. In these classrooms, the pupils find the stage continually set to arouse their feeling of need for a particular arithmetic skill or process, then drill and exercise follow to fix permanently these learnings. This point of view holds certain selected arithmetic content to be so valuable socially that activities must be found or stimulated in order to motivate the work. The arithmetic content is the immediate end-point and the activity becomes the method of teaching it. Formal systematic daily arithmetic which follows a course of study or textbook governs the selection and initiation of activities.

A third viewpoint: recognizing two goals. There is still another—a third—point of view concerning this issue and the majority of the members of the committee responsible for this report subscribe to it. To some extent it is a synthesis of the two positions previously discussed. In the first place, in the classroom of those adhering to this third position, the aim is to develop in children those interests and urges which seem most worthwhile when all things are considered, i.e., no large activity is selected solely because it offers unusual opportunities for arithmetic. In the full development of most activities, as is illustrated in the case of the study of aviation, arithmetic will be necessary many times. Here arithmetic serves as a tool or as a means to the solution of the larger purpose, rather than as an end in itself. Arithmetic is a part of the process by which the child refines—resolves his problem. There is no forcing of artificial experiences in order to make opportunities for arithmetic, but whenever the situation functionally calls for arithmetic solution, it naturally comes into the activity. Furthermore, whenever the arithmetic process or skill is a new one to the children or has not been adequately fixed in their neural patterns, time is set aside in arithmetic periods for practicing these aspects of the subject. (The intelligent teacher, of course, passes over a process which she believes too difficult for the pupils to master or which is not apt to come up again until several grades later in the school career. She will need to check such judgments against standards held in her school system.) The skill or process is first called into meaning by the demands of the problem and then effort is devoted to the fixation of the process or skill through utilizing the best methods of learning known to the teachers.

This point of view further recognizes that there are undoubtedly certain aspects of arithmetic which can best be learned by a systematic practice when the learner has once grasped the meaning of a process. Also, many arithmetic manipulations depend upon a mastery of more elemental processes—multiplication can be done when one can add figures, etc. Much of the mathematics work in the high school is predicated on the mastery of certain skills in the lower grades. A mobile population adds to the desirability of some common consent among schools concerning the placement of the major learnings. All these considerations presume a need for some minimum understanding and facility of quantitative thinking and some minimum skill in fundamental processes.

This position differs from the first position presented in this paper in denying that functional experiences of childhood are alone adequate to develop arithmetic skill. On the other hand, this committee does not agree with the usual program of systematic mastery goal. The committee feels that their experience in teaching children has shown them that the traditional course of study is faulty in its selection of skills to be learned and in their grade placement. This committee holds that the teaching of arithmetic must take account of the following:

1. The demonstrated greater effectiveness of learning which results from the pursuit of a meaningful and purposeful activity.
2. The maturation of the insight and the interests of children at various ages.
3. The "logical" development of many aspects of arithmetic.

A survey undertaken. In order to find out more definitely whether the viewpoint held by the members of this minority group is sound, a survey was next undertaken in order to discover the extent to which opportunities for arithmetic in the activities program were possible. Six teachers of Grade 3 and six teachers of Grade 6 were invited to coöperate. These teachers were chosen from public and private schools in New York City, from suburban communities adjacent to New York City, and from rural communities of New Jersey. The necessity for group meetings to discuss survey techniques made it unwise to extend the survey widely throughout the United States in this initial undertaking. The two levels, Grades 3 and 6, were selected in order to sample the elementary school. These schools for the most part teach some formal arithmetic, but on the whole these schools are organizing and integrating the children's experiences around worthwhile interests rather than following a strictly separate subject approach with a systematic course of study.

Each teacher was asked to record on prepared blanks every situation faced by individuals or by her entire class, in which there was a need for quantitative thinking and manipulation. These arithmetic situations were not to be those suggested by arithmetic courses of study, textbooks, or workbooks, but those problems which arose in the pursuit of some child-selected activity. An effort was made to explore the out-of-school arithmetic experiences whenever possible. For the most part, however, the recorded problems grew out of class activities rather than individual problems of a nonschool nature.

The recording blank (only one problem to each blank) called for the following:

1. The situation—a brief statement of the unit of work or the activity in which the demand for the use of arithmetic arose.
2. The arithmetic problem—the problem worded in terms similar to the usual textbook statement with all essential numbers.
3. Computation involved in solution (to be filled in by the chairman of the group).
4. Space provided for teacher's name, grade, date, and number of children who faced this problem.

A typical completed record sheet for the third grade and one for the sixth grade follow:

Opportunities for the Use of Arithmetic in an Activity Curriculum

The situation: [A brief statement of the unit of work or the activity in which the demand for the use of arithmetic arose.]

The fire house that John and Joseph are building for the community study is to be painted. The following articles are needed:

The arithmetic problem: [The problem to be worded in terms similar to the usual textbook statement with all essential numbers.]

1 Can of enamel	15¢
1 Can of flat paint	10¢
2 Brushes at 5¢ each	
Total cost	?

Computation involved:

$$\begin{array}{r} 5\text{¢} \\ \times\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 15\text{¢} \\ 10\text{¢} \\ 10\text{¢} \\ \hline \end{array}$$

Teacher Grade 3 Date

Opportunities for the Use of Arithmetic in an Activity Curriculum

The situation: [A brief statement of the unit of work or the activity in which the demand for the use of arithmetic arose.]

The situation in which this problem arose was in a unit entitled: "The Solar System—How Our Earth Became a Member Planet."

The arithmetic problem: [The problem to be worded in terms similar to the usual textbook statement with all essential numbers.]

A light year is the distance light travels in one of our years at the rate of 186,000 miles a second (approximate). This is the unit scientists use in measuring the distance of the stars. What is the distance of a light year in miles?

Computation involved:

186,000	11,160,000	66,960,000	1,607,040,000
× 60	× 60	× 24	× 365

Teacher Grade 6 Date

Twelve teachers kept such records of arithmetic needs for the last two months of the spring term and the first two months of the fall term of 1933. The number of specific problems recorded totaled 439; 234 for Grade 3 and 205 for Grade 6. The majority of these problems arose in connection with a few large units of work. For example, one of the units of work in Grade 3 dealt with the "Story of Old New York." Among the arithmetic problems connected with this unit were: the making of individual booklets (involving size of cover, size of end sheets, size of binding tape, number of sheets of paper, etc); making hats for the dramatization (size of each head, using string and pencil compass, etc.); making a beaverboard sailboat for the dramatization (number of persons to ride in boat, necessary boat size, measuring beaverboard, measuring sail, etc.); and making a frieze of Old New York (measuring wall space available, counting number of scenes to be painted, dividing space by number of scenes, etc.).

One of the units of work in Grade 6 dealt with the "Story of the Solar System and the Beginnings of Our Earth." Among the problems arising were: determining the size of prehistoric animals from scale drawings (measuring size of drawing, multiplying by scale value); computing the time for an airplane to travel to the moon (dividing distance by speed of plane, dividing hours by 24, dividing days by 7 in order to express time in weeks); making a planetarium (finding relative distances of planets, relative size, agreeing on manageable scale, etc.); constructing a time chart (figuring relative length of various ages, measuring length of beaverboard, determining scale, laying out time line on board, etc.); determining how many block prints of prehistoric animals are necessary for individual booklets. (Each child made one animal print, therefore the number in the class times the number of different prints, plus extras for possible waste, etc., constituted the problem.)

Some of the problems recorded came in connection with individual or with group activities in no way connected with the major units of work. For instance, a child in Grade 3 had to leave school at eleven-thirty two days a week for a special music lesson. It was necessary for the pupil to watch the clock on those days and to depart on time. In Grade 6 one group had health inspections. The children computed the percentages of the pupils having good posture, hearing, sight, etc., in order to note growth.

Analysis of the problems. The 439 problem sheets were studied in order to find answers to the following questions:

1. How many one-step, two-step, and three-or-more-step problems do children solve in connection with their activities?
2. How many operations in each of the four fundamental processes were completed?
3. How many solutions involved integers, fractions, mixed numbers, decimals, decimal-fractions, linear measure, time, calendar, etc.
4. How many problems demanded such miscellaneous manipulations as measuring, comparison, counting, reading and writing numbers, telling time, graphing, etc.
5. What was the general nature of the arithmetic problem arising in Grade 3? In Grade 6?

Types of problems in Grade 3. The majority of the problems recorded in the third grade classes arose in connection with seven units of work. A brief statement of the theme of each unit is given here with a list of its problem situations.

Story of Old New York. This was a study of the early history of the Manhattan Indians who slowly gave way to the Dutch who, in turn, gave way to the English. During this transition period New York City came into existence. The study culminated in a historic frieze, booklets, and dramatizations.

Estimating age of modern means of travel in New York.

Figuring age of the United States as an independent country.

Measuring wall space for a historic frieze, dividing it into scenes, allocating space to each.

Measuring beaverboard for boat to be used in dramatization. Cutting to measure.

Measuring head circumferences for paper hats, width of brims, laying out pattern with compass.

Figuring size of cover and sheets to use in individual booklet, estimating number of sheets needed, binding tape needed, etc.

Food Study. The children undertook a study of the foods eaten by their class, the sources, effects on health, etc. In connection with the study were many experiences in cooking in the Household Arts kitchen.

Timing the freezing of sherbet.

Counting number of children to be served spaghetti.

Weighing a quart of milk and estimating weight of a gallon.

Finding number of tablespoons in cup.

Using ½, ¼, and ⅔ cup.

Figuring cost of homemade butter and comparing cost with butter bought at store.

Understanding of statement that 60% water is evaporated in condensed milk.

Doubling a recipe containing 3 cups, ½ cup, ⅓ cup, 1½ cups.

Counting number of children, cups, spoons, dishes, napkins.

Taking ½ of recipe.

Adding up the cost of a list of groceries and getting change from $1.00.

Using candy thermometer at 280°.

Multiplying lunch price by number of persons who paid, then figuring profit.

Checking on change from $1.00.

Checking on change from 50¢.

Adding cost of items at cafeteria and checking on change from 50¢.

Adding amounts earned by children during month for purchase of milk.

Cost of milk for month, days, and cost per quart known.

Change from 25¢, 50¢, and $1.00 for 15¢ purchase.

Finding how many bottles of milk at 5¢ each a one-dollar bill will purchase.

Figuring change from 5¢ in purchase of 3¢ stamp for letter to be sent to bottling plant.

Finding monthly milk bill for class.

New York, The Wonder City. The pupils surveyed their city's water supply, harbor, police, fire department, sanitation, etc.

In an hour period, how much time would be spent en route and how much in observation at fire house? Time for leaving fire house.

Figuring lumber needed to build small fire house in classroom.

Finding out how many sections of a 15-foot ladder would be needed to reach roof of various houses.

Finding out number of feet and yards of fire hose to reach from hydrant to school house.

Comparing cost of nails from several hardware stores. Finding number of pounds of nails which can be purchased for 25¢.
Finding out number of days to elapse until dramatization.

The Story of Clocks. This unit involved a series of activities with sand, water, and fire clocks, as they were invented and improved by men of early times.

Measuring water and sand in pints, quarts, gallons.
Checking sand, water, and fire clocks with standard timepiece.
Comparing length of candles.
Translating minutes into fraction of hours. Reverse process.
Measurement of candles in fractions of inches.
Increasing sand or water to make a clock of specified time period.
Measuring lapse of time.
Meaning of Roman numerals on clock.

Beginnings of the Earth. Another unit of work was a study of the solar system, its history, the earth in its larger setting, life on the earth, prehistoric animals, etc.

Writing and reading large numbers, distance, and time lapse.
Comparison of distances to planets and stars—farthest, nearest.
Comparison of sizes of planets—largest, smallest, etc.
Comparison of diameter and circumference of earth.
Contemplation of earth's age (600,000,000).
Estimating lapse of time for various life changes on earth.
Speculation of largest number of stars possible.

Life in Anne Hutchinson's Time. A study of the life of this period and the hardships endured in achieving modern life revived crafts, songs, games, and dances of the time. The children made an old-fashioned garden on the school roof.

Counting letters in sampler.
Counting spaces between letters.
Counting spaces between words.
Counting spaces in margins.
Computing length and width of sampler.
Estimating size of wood for rockers for cradle.
Lapse of time—Hutchinson to present.
Electing officers for publication—counting ballots, marking off into groups of five, comparing closeness of votes.
Finding number of small pieces of wood which can be cut from a large board.

Estimating number of wooden nails three children can cut.
Estimating size of printed page after margins are taken out.
Cost of bus ride per child.
Number of additional children needed to pay for bus trip.
Estimating size of cloth for bed cover and pillow for doll bed.

Life in Holland. This was a study of the people who live in Holland: their home life, transportation, production, holidays, etc.

Increased dollar value of a Rembrandt painting.
Average number of pages per chapter in story book.
Total money earned by class.
Comparison of two totals of money earned.
Total weight of tray of cheese.
Translation of dozen into units.
Total weight of cheeses.
Earning per week in cheese-making.
Lapse of time since tulip was introduced into Holland.
Difference in time in Christmas here and in Holland.
Total cost of Christmas presents for Dutch child.
Lapse of time since Rembrandt painted his portrait.
Comprehension of two heights, one 40 feet and one 150 feet.
Measurement of length of room.
Comparison of 40 feet with known length of room.
Doubling a large number.
Finding working space on page after margins are blocked off.

The remainder of the problems recorded arose in connection with telling time and with general classroom housekeeping activities. A few non-unit problems which originated in out-of-school experiences of the pupils.

Telling time for next class.
Understanding terms of half-past and quarter-past.
Leaving classroom for assembly at appointed time.
Comparison of New York time with European countries and western states. Use of radio in this connection.
Finding number of minutes until noon.
Finding one-quarter of a dollar is not same figure as one-quarter of an hour.
Time until vacation.
P.M. and A.M. concepts.
Counting children present and absent.
Dividing class into two groups.
Measuring length of room.

Verifying best authority from copyright dates.
Quickly finding page in reading book.
Totaling lunch order.
Cost of number of tickets.
Making change in selling tickets.
Counting to 20 to check tickets sold.
Checking total tickets against those sold.
Final accounting of money and tickets.
Estimating number of people coming to party.
Determining size of batches of cookies and punch to serve 60 people.
Determining total cost of party.
Measuring size of paper and lumber.
Figuring number of shelves needed.
Placement of cupboard partitions.
Measuring for screw holes.
Measuring size of covers.
Figuring cost of notebooks for class.
Inventorying room supplies and equipment.
Reading temperature of room thermometer.
Reading time of clock to take thermometer reading.
Lowest temperature.
Highest temperature.
Change in temperature daily.
Greatest and least temperature daily.
Period of greatest gain in temperature.
Measuring classroom size and reproducing it in scale drawing.
Finding cost of 10 typewriters in room.
Purchasing bulbs for school garden.
Gain or loss in weight since last measurement.
Gain in height since last measurement.
Lapse of time for postcard to come to New York from India.
Comparison of age of coins in a collection.
Estimating number of pipes and stops on an organ.
Establishing priority of one of two historic events.
Comparing the dirigible Akron's 207 passengers with known number of persons in school assembly.
Number of $\frac{1}{3}$'s in 1.
Adding up bill of materials.
Cost of feed for bird.
In making dictionary, counting letters in alphabet, deciding number of pages to each letter in 80-page notebook.
Determining amount of material for window curtains.
Learning about half, quarter, and eighth notes in music.

Contemplating how far 5 miles would be.
Placing Declaration of Independence and end of Revolution.
Computing number of United States senators.
Figuring number of United States representatives from New York and California.

Types of problems in Grade 6. In the problems recorded for pupils of Grade 6, the majority were found in connection with ten units of work. Following are brief statements of these units with the number situations involved:

Study of New York's Water Supply. A class went to Kensico Dam to study the source of New York's water supply.

Recording the daily contribution of the pupils to cost of bus trip.
Figuring distance of bus trip.
Estimating per-pupil cost of trip.
Figuring fraction of total trip covered at various stopping points.
Converting water in dam to quarts.
Figuring percentage one number is of another.
Finding what percentage of the water is purified.
Estimating gallons of water used in month.
Getting total costs of bus, lunch, etc.
Finding amount of chlorine used in a day for purification.

The Solar System and Life Beginnings. This was a study of the sun, planets, and other solar systems. They developed the story of how life began and increased on the earth.

Translating scale drawing of animals to proportions for clay models.
Computing time for trip to moon in airplane, traveling 300 miles per hour.
Relation of diameter to circumference.
Drawing to scale distances and sizes of planets and sun.
Constructing planetarium.
Constructing time chart of ages of earth.
Estimating amount of paper for class booklets.
Estimating orbits of planets with diameters known.
Estimating number of times light travels around earth in one second.
Comparing speed of train and light.
Reading large numbers.
Computing number of miles light travels in a year.

Health Work. A study was made of vitamins, diet, etc., necessary to growth. The children experimented on diets with animals. The health of the class was checked.

Making a graph of growth of two cages of rats fed different diets, involving weighing and averaging.

Figuring percentage of class members having standard sight, hearing, etc.

Natural Wealth of the United States. A study was made of oil, water, climate, soil, technology, etc., in relation to the wealth of the United States.

Ordering small desk maps with check enclosed for payment.
Graphing increase in oil production since 1900.
Graphing water power of United States and comparing it with foreign countries.
Finding out the proportion of cars to population.
Comparison of rainfall maps.
Translating scale into miles from map.

Hallowe'en. Activities in connection with preparation for Hallowe'en celebration were as follows:

Determining best price on masks and pumpkins for entire class.
Finding total cost of pumpkins.
Finding profit from sale of pumpkins.
Estimating loss from spoiled pumpkins.
Computing total profits from sale.
Dividing profit into two funds.
Estimating profit from sale of large number of items at festival.

A School Store. A class kept for sale certain articles which were used by the school.

Finding cost of shipment of paints and paper.
Determining selling prices.
Estimating profits.
Borrowing to pay for shipment.
Figuring interest on loan.
Balancing account each month and paying off loan.
Finding cost of second shipment at increased new price.
Figuring new selling price.
Figuring new profits.

Study of Carelessness. The pupils figured what breakage and lost articles cost the school annually.

Finding cost of replacing brushes, scissors, screw-drivers, hammers, planes, saws, chisels, files, pliers, awls, clamps, knives, balls, and books.
Adding the total bill.

Anne Hutchinson's Time. A study was made of the clothing worn when Anne Hutchinson was living:

Determining cost of materials needed in unit.
Finding balance in funds.
Comparison of prices of muslin at various stores.
Determining amount of cloth needed to dress dolls and make costumes for play.
Comparison of time lapses from Hutchinson's time to present.
Age of Anne at marriage.
Age of Suzanne when returned by Indians.
Time in America when banished.
Age of Anne at death.
Construction of model house.
Scale drawing of house.
Scale drawing of furniture of house.
Estimating cost of excursion.
Sharing cost among class.
Selecting meal from menu card with limited funds available.
Collecting money from children at table to pay check and making correct change for each child.
Laying out pages of class booklet.
Counting links in crochet and finding center of pattern for making small designs.

Western Movement. The purchases of land which made the western territory accessible to our citizens was particularly emphasized in a unit of study concerning the Western Movement.

Computing amount of money spent for land over budget figure. What percentage?
Translating square miles into acres.
Computing price per acre.
How many acres can be purchased for $15,000,000?

Club Activities. A group of boys having a checking account purchased baseball equipment.

Purchasing bat at discount. Compute actual cost.
Figuring discount allowed on ball.
Comparing net cost with price at another store.
Writing a check for purchases.
Balancing checkbook.

The remainder of the recorded problems came in isolated activities in and out of school. They are as follows:

Finding how many admissions at 15¢ will total $14.25.
Estimating profit from sale of snapshots.
Measuring size of bulletin board, dividing into sections, one for each pupil to make a scene for frieze on China.
Computing age of Confucius.
Computing amount of beaverboard needed to build Greek temple for classroom.
Making a time line of Greek history.
Computing money value of scrip held by class.
Averaging term tests.
Apportioning cost of flowers among class members.
Budgeting cost of materials for Memorial Day play.
Spacing exhibit on bulletin board.
Spacing plants in border of garden.
Computing distance of Arcturus from earth. Comparing with earth-to-sun distance.
Adding incomes from various tables where food was sold.
Computing average speed of Lindbergh in Paris flight.
Comparing land and water surface on earth.
Comparing speed of men, horse, train, auto, and airplane.
Measuring windows and purchasing materials for curtains.
Finding center of paper for typewriting.
Measuring bricks for Pueblo.
Plotting music practice period over three weeks.
Computing how many quarts of milk can be bought with funds available.
Figuring total cost of subscriptions to newspaper for members of class.
Computing cost of photographic supplies and apportioning among four children.
Apportioning books among sections of sixth grade.
Purchasing groceries and making change.
Figuring amount of material for dress and purchasing it.
Purchasing clothing.
Translating speed at sea (knots) into speed on land.

Types of problems. An analysis of Table I shows that the pupils in Grades 3 and 6 found 439 problems growing out of their activities. Of these, 279, or 63%, involved computation while 160, or 37%, involved no computation. The pupils in Grade 3 found that 56% of their problems involved computation while the pupils in Grade 6 found that 72% of their problems were classified as computation. This higher percentage of computational problems in the upper grade is worthy of note. The problems are 44% non-computational for Grade 3 as against 28% for Grade 6.

A further analysis of Table I discloses that of a total of 279 computational problems for the two grades combined, 145 problems, or 52%, are one-step problems; 72, or 26%, are two-step problems; and 62, or 22%, are three-or-more-step problems. When the two grades are compared, it is seen that 72% of the Grade 3 problems are one-step as against 34% of the Grade 6 problems; 16% of Grade 3 problems are two-step as against 34% of Grade 6; 12% of Grade 3 problems are three-or-more-step, while 31% of Grade 6 problems are of this more complex type. This difference in the percentage of simple one-step problems in the two grades is marked.

The four fundamental processes. The 279 problems involving computation necessitated 640 computational manipulations for solution. Table II shows that 194 computations were carried out in Grade 3 and 446 in Grade 6. Of the total for the two grades 24% were in addition, 23% were in subtraction, 38% were in multiplication, and 15% were in division. In Grade 3, 19% were in addition, as compared with 26% in addition in Grade 6. Subtraction seems to be more frequent in Grade 3 where 36% of all problems were in subtraction, as compared with 17% for Grade 6. Multiplication accounts for a disproportionate number of computations both in Grade 3 (30%) and in Grade 6 (41%). Division is the least used of the four processes in both grades—14% in Grade 3 and 15% in Grade 6.

Types of numbers. A study was made of the type of numbers used in these problems. Table III shows that of the 640 computations completed in the two grades, a total of 288, or 45%, were integers, 8, or 1%, were fractions, 43, or 7%, were mixed numbers, 277, or 42%, were decimals (almost entirely money), while decimal fractions and other types accounted for 1% and 3% of the total. The high frequency of integers and decimals is outstanding. In Grade 3 integers and decimals (money) make up 65% and 17%, or 82%, of the total computations for the grade. Likewise integers and decimals (money) make up 36% and 54%, or 90%, of the total for the sixth grade. The comparison between integers in Grade 3 (65%) and in Grade 6 (36%) shows a tendency away from whole numbers as the pupils face more complex problems in the upper grade and correspondingly more computation involving decimals, Grade 3 showing 17% and Grade 6 54%. The slight decrease from

TABLE I

NUMBER OF PROBLEMS INVOLVING COMPUTATION AND NUMBER OF PROBLEMS LACKING COMPUTATION, WITH NUMBER OF STEPS NECESSARY FOR SOLUTION AND PERCENTAGES

GRADE	PROBLEMS INVOLVING COMPUTATION								PROBLEMS LACKING COMPUTATION		TOTAL
	One-Step		Two-Step		Three- or More-Step		Total	% of Total Involving Computation	No.	% of Total Lacking Computation	
	No.	%	No.	%	No.	%					
III	94	72	21	16	16	12	131	56	103	44	234
VI	51	34	51	34	46	31	148	72	57	28	205
Total	145	52	72	26	62	22	279	63	160	37	439

[*Table II on page 104*]

TABLE III

TYPES OF NUMBERS FOUND IN THE PROBLEMS OF GRADES 3 AND 6, WITH PERCENTAGES

GRADE	INTEGERS		FRACTIONS		MIXED NUMBERS		DECIMALS		DECIMAL FRACTIONS		OTHER*		TOTAL
	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	
III	126	65	7	4	17	9	33	17	1	0	10	5	194
VI	162	36	1	0	26	6	244	54	6	1	7	2	446
Total	288	45	8	1	43	7	277	42	7	1	17	3	640

* Other types consist of calendar computation, time computation, and linear computation.

Grade 3 to Grade 6 in percentage of computations involving fractions, mixed numbers, and other types is probably too small to be significant.

A study of integers. Table IV shows the classification of integers according to the fundamental processes. The last column in the table shows a fairly even distribution of the four processes.

TABLE II

NUMBER OF COMPUTATIONS INVOLVING EACH OF THE FOUR FUNDAMENTAL PROCESSES, WITH PERCENTAGES

GRADE	ADDITION		SUBTRACTION		MULTIPLICATION		DIVISION		TOTAL
	No.	%	No.	%	No.	%	No.	%	
III	38	19	70	36	59	30	27	14	194
VI	115	26	78	17	185	41	68	15	446
Total	153	24	148	23	244	38	95	15	640

Marked differences are seen between the two grades in subtraction, only 5% of integers in Grade 6 as against 39% in Grade 3. To a lesser degree a difference shows up in addition, 14% for Grade 3 as against 37% for Grade 6. Both grades show about one-fourth of all integer computations to be multiplication.

TABLE IV

THE DISTRIBUTION OF THE 288 COMPUTATIONS INVOLVING *Integers* FOUND AMONG THE FOUR FUNDAMENTAL PROCESSES, WITH PERCENTAGES

GRADE	ADDITION		SUBTRACTION		MULTIPLICATION		DIVISION		TOTAL
	No.	%	No.	%	No.	%	No.	%	
III	18	14	50	39	35	28	23	18	126
VI	60	37	9	5	41	25	52	32	162
Total	78	27	59	20	76	26	75	26	288

A study of decimals. Table V indicates that 50% of the 277 computations of decimals are fairly evenly divided between addition and subtraction. Multiplication accounts for 46%, while division is seldom used in decimals, only 4% falling in this classification. More than a majority of all decimal computations in Grade 3 deal with subtraction, but it must be noted that the total number of decimal operations in Grade 3 is only 33. Any conclusion based on these facts is necessarily tentative. There are no decimal computations for division in Grade 3. In Grade 6 multiplication of decimals accounts for 50% of all of this classification for this grade. A later portion of this study will show that these decimal operations were for the most part money transactions -

TABLE V

THE DISTRIBUTION OF THE 277 COMPUTATIONS INVOLVING *Decimals* FOUND AMONG THE FOUR FUNDAMENTAL PROCESSES, WITH PERCENTAGES

GRADE	ADDITION		SUBTRACTION		MULTIPLICATION		DIVISION		TOTAL
	No.	%	No.	%	No.	%	No.	%	
III	6	18	18	55	9	27	0	0	33
VI	50	20	59	24	122	50	13	5	244
Total	56	23	77	27	131	46	13	4	277

adding items in a bill of expense, finding change due in a purchase, or multiplying a price by the number of articles to be purchased.

Other types of numbers. Table III also shows that fractions (1%), mixed numbers (7%), decimal fractions (1%), and others (3%) represent such a small part of the total that there is no need for a closer analysis. The important point to be noted is the infrequent use pupils of Grades 3 and 6 make of these types of numbers as compared to integers and decimals (money).

TABLE VI

CLASSIFICATION OF THE 160 NONCOMPUTATIONAL NUMBER SITUATIONS, FACED BY GRADES 3 AND 6

	Grade III	Grade VI	Total
Measuring	50	9	59
Comparison	18	15	33
Counting	23		23
Making pattern	1		1
Reading large numbers	1		1
Writing large numbers	1		1
Telling time	3		3
Graphing		4	4
Scale drawing		5	5
Other types	6	24	30
Total	103	57	160

The problems lacking computation. Table I indicates that a total of 160 problems recorded lacked computation. A further analysis of these problems is shown in Table VI. Measuring, comparison, and counting constitute over two-thirds of all these problems. Graphing and scale drawing in Grade 6 occur often enough to merit mention. Also, the noncomputational problems are twice as numerous in the lower as in the higher grade.

CHART I

Problems in *Addition* Actually Done by Pupils in Grade 3

Integers 2 numbers	9 + 12 420 + 25 120 + 100	75 + 45 1 + 4 5 + 6	74 + 33 15 + 10 12 + 3
3 numbers	27 + 7 + 33 10 + 10 + 8 15 + 10 + 10		10 + 10 + 10 6 + 12 + 8
4 or more numbers	5 + 10 + 10 + 10 + 10 20 + 20 + 20 + 10 + 20 98 + 48 + 22 + 12 2 + 1 + 2 + 2		
Fractions 2 fractions	½ + ½	¼ + ¼	⅛ + ⅛
4 or more fractions	¼ + ¼ + ¼ + ¼ ⅛ + ⅛ + ⅛ + ⅛ + ⅛ + ⅛ + ⅛ + ⅛ ⅛ + ⅛ + ⅛ + ⅛		
Mixed Numbers 3 numbers	2 + ½ + ½ 3½ + ½ + ½		3½ + ½ + ½ 12 + ½ + 1
Decimals* 2 numbers	.65 + .65 2.00 + .50 7.00 + 4.25		10.35 + .05 2.50 + 2.75
3 numbers	5.00 + .65 + .75		
Other Types Linear Measure	3 yd. 2 in. + 12 in. 24 yd. + 3 yd. 4 in.		
Time	10:55 + 5 min. 11:00 + 2 min.		

* All figures.

A study of addition in Grade 3. Chart I presents a study of addition in Grade 3, showing the specific combinations utilized and the complexity of the computations. With integers there is a fairly even distribution among addition of two numbers, three numbers, and four or more numbers. Many of the additions can be made without carrying. Only two of the problems involve numbers in the hundreds, while only five add up to sufficient totals to be in the hundreds. Many of these figures are sums of money.

With addition of fractions and mixed numbers, only the fractions ½, ¼, and ⅛ are needed. In decimals all the figures represent money transactions. Other types include addition of yards and inches and addition of hours and minutes.

CHART II

PROBLEMS IN *Subtraction* ACTUALLY DONE BY PUPILS IN GRADE 3

Integers	10 − 3	25 − 3	14 − 10
	12 − 10	12 − 10	12 − 11
	15 − 9	5 − 3	38 − 15
	17 − 4	22 − 4	50 − 10
	60 − 50	28 − 26	25 − 5
	40 − 30	22 − 2	72 − 68
	78 − 68	73 − 69	78 − 11
	78 − 71	71 − 68	74 − 71
	75 − 70	76 − 72	100 − 26
	6 − 2	30 − 23	20 − 6
	20 − 3	120 − 100	30 − 25
	25 − 21	25 − 24	42 − 11
	1933 − 1643	1933 − 1675	1933 − 1640
	1933 − 1300	1781 − 1776	1933 − 1826
	1933 − 1897	1933 − 1907	1933 − 1920
	1933 − 500	1492 − 1432	1933 − 1829
	1933 − 1776	350,000 − 30	
Decimals (all money)	1.00 − .69	1.00 − .30	1.00 − .30
	.50 − .30	.50 − .25	.25 − .15
	.50 − .15	.20 − .15	.25 − .15
	.50 − .30	.50 − .30	.50 − .15
	1.00 − .15	1.00 − .15	.35 − .30
	1.00 − .50	6.00 − .69	.30 − .30
Other Types	11:30 − 5 min.	5/25/33 − 4/16/33	

A study of subtraction in Grade 3. Chart II lists the actual subtractions made in Grade 3. Most subtractions of integers are here shown to be of two-place numbers, the major exceptions are those involving dates. Only one large number is shown. As was the case with addition, the subtraction of decimals is entirely involved in money problems. There were no problems involving subtraction of fractions or mixed numbers. Calendar-computation and time problems constitute the two listings under other types.

A study of multiplication in Grade 3. Chart III shows that integers are more frequent than other types of numbers. There is a fairly wide sampling of the multiplication facts although most of the facts in the multiplication tables are not involved in the problems recorded. Fractions are confined to ½, ⅓, and ¼. The only case in all Grade 3 problems where the decimal fraction is used is found here in multiplication. As has been true in addition and subtraction, decimals were entirely confined to money problems.

CHART III

PROBLEMS IN *Multiplication* ACTUALLY DONE BY PUPILS IN GRADE 3

Integers	1 × 2 2 × 3 2 × 4 2 × 6 2 × 26 3 × 6 3 × 60 5 × 10 5 × 20 6 × 180 10 × 40 15 × 30	1 × 12 2 × 3 2 × 5 2 × 6 2 × 48 3 × 11 3 × 100 5 × 12 5 × 100 7 × 14 13 × 18 20 × 21	2 × 2 2 × 4 2 × 5 2 × 11 2 × 200 3 × 26 4 × 12 5 × 20 6 × 10 9 × 10 15 × 20
Fractions		½ × ⅔	
Mixed Numbers	⅓ × 36 ½ × 2 ½ × 100 3 × 4½	¼ × 100 ½ × 2 ½ × 4 1½ × 2	¼ × 60 ½ × 2 ½ × 9 4½ × 13½
Decimals (money)	1 × .25 3 × .50 20 × .30	2 × .25 14 × .50 20 × .30	2 × .50 17 × .50 20 × .30
Decimal Fractions		.12 × .27 1/9	
Other Types		8 × 3 yd. 14 in.	

A study of division in Grade 3. Chart IV shows that practically no division is needed in Grade 3 problems outside of integers. Short division without remainders has the same number of computations as long division. Many of the long-division problems could be made into short-division problems by dividing by 10 or multiples of 10. The computations of short division with remainders are fewer in number than the other two classifications.

A study of addition in Grade 6. Chart V shows a large number of additions of integers and decimals with only one computation in linear measure and four computations in mixed numbers. Most of the integers are one-place numbers. The figures *1* and *5* are most frequently used due to the fact that these computations were additions of sums of money. The "penny" and "nickel" are evidently familiar coins among the pupils. A group of additions of 80's and 90's, involves the averaging of school marks.

CHART IV

PROBLEMS IN *Division* ACTUALLY DONE BY PUPILS OF GRADE 3

Integers Short No Remainder	3 ÷ 1	8 ÷ 2	150 ÷ 2
	600,000,000,000 ÷ 2	16 ÷ 4	25 ÷ 5
	7 ÷ 7	12 ÷ 1[illegible]	60 ÷ 10
Short Remainder	9 ÷ 2	23 ÷ 8	25 ÷ 7
	30 ÷ 7	31 ÷ 7	190 ÷ 6
Long	50 ÷ 25	100 ÷ 20	200 ÷ 50
	150 ÷ 30	112 ÷ 36	500 ÷ 50
	2400 ÷ 8000	12,633,000 ÷ 280,000	
Mixed		1 ÷ ⅕	
Unclassified		h ÷ 15	
		l ÷ 3	
		80 ÷ ?	

As in Grade 3, the fractions are all simple—½, ¼, ¾, ¹⁄₁₂. The decimals for the most part refer to money transactions. There is some question as to whether these figures should not have been set down as integers rather than decimals. Pupils probably think of "small change" as units of cents rather than as decimal parts of a dollar. However, the teachers recording the problems indicate that the pupils were in these instances using decimals to compute the answer.

A study of subtraction in Grade 6. Chart VI lists the computations made in subtraction, the majority of which consist of

CHART V

Problems in *Addition* Actually Done by Pupils of Grade 6

Integers		
2 numbers	1933 + 550	
3 numbers	10 + 10 + 10	10 + 10 + 5
4 or more numbers	1 + 1 + 2 + 5 + 3	
	5 + 1 + 5 + 1 + 5 + 2	
	3 + 1 + 2 + 1 + 5 + 1	
	1 + 1 + 5 + 1 + 2 + 10 + 1 + 1 + 1 + 2 + 3 + 1 + 1 + 4 + 2 + 3	
	1 + 10 + 10 + 5 + 5 + 5 + 1 + 1 + 5 + 1 + 5	
	5 + 5 + 5 + 5	
	1 + 1 + 1 + 1	
	5 + 1 + 5 + 5 + 2 + 5	
	1 + 5 + 5 + 1 + 5 + 1 + 2	
	1 + 1 + 1 + 10 + 5 + 5 + 1	
	1 + 1 + 1 + 1 + 5 + 1	
	1 + 1 + 1 + 10 + 5 + 10	
	2 + 2 + 1 + 5 + 1 + 5 + 2	
	5 + 1 + 5 + 1 + 1 + 5	
	3 + 1 + 5 + 1 + 2 + 2 + 5 + 5	
	1 + 5 + 5 + 2 + 5 + 5	
	1 + 2 + 2 + 5 + 2 + 1 + 1 + 5	
	1 + 1 + 2 + 1 + 1 + 1 + 5	
	1 + 1 + 2 + 1 + 2 + 1 + 1 + 1 + 1 + 2 + 6 + 3 + 1 + 5 + 1 + 1	
	5 + 10 + 5 + 10 + 25	
	5 + 5 + 5 + 5 + 5 + 5 + 5	
	1 + 1 + 1 + 1 + 1	
	5 + 5 + 5 + 5 + 5 + 5	
	10 + 10 + 10 + 10 + 10 + 10	
	5 + 3 + 10 + 5	
	5 + 5 + 1 + 2 + 1 + 1	
	1 + 5 + 5 + 1 + 5 + 5 + 2 + 1 + 1	
	1 + 1 + 1 + 1 + 2 + 1	
	4 + 1 + 1 + 5 + 2 + 1 + 1	
	1 + 1 + 1 + 1 + 5 + 2	
	1 + 2 + 3 + 3 + 5 + 5	
	2 + 3 + 1 + 5 + 1 + 1 + 2 + 1 + 1 + 5 + 1 + 1	
	1 + 1 + 4 + 1 + 1 + 5 + 1	
	2 + 5 + 5 + 10 + 2 + 5 + 1 + 1 + 1 + 2 + 1	
	1 + 1 + 4 + 1 + 1 + 5 + 1	
	2 + 5 + 5 + 10 + 2 + 5 + 1 + 1 + 1 + 2 + 1	
	1 + 1 + 1 + 1 + 5 + 5 + 1	
	1 + 5 + 10 + 5 + 10 + 5	
	1 + 10 + 5 + 1 + 10 + 1 + 10 + 1 + 5	
	1 + 10 + 5 + 5 + 1 + 1 + 2 + 2 + 3 + 5 + 1 + 2	
	2 + 5 + 5 + 10 + 5 + 5 + 1 + 5 + 2 + 1 + 1 + 1 + 1	
	6 + 10 + 1 + 5 + 10 + 5 + 5 + 5 + 1 + 5 + 1 + 3 + 1 + 5	
	1 + 5 + 10 + 2 + 1 + 5 + 5 + 1 + 5 + 10 + 1 + 1 + 2 + 5 + 5 + 5 + 5	

CHART V (*Continued*)

Integers (*Continued*)	1 + 25 + 10 + 5 + 5 + 5 + 5 + 1 + 2 + 5 + 5 + 5 + 5 + 1 + 1 + 1 + 1 + 2 + 5 + 3 + 2 + 1
	30 + 49 + 24 + 14 + 35 + 14 + 36 + 33 + 38 + 44 + 63 + 65 + 53 + 56 + 69 + 96
	30 + 58 + 5 + 18 + 40 + 35 + 25 + 5 + 30 + 27 + 60 + 20 + 5 + 23 + 20 + 24 + 10 + 23 + 18 + 24 + 23 + 20 + 12 + 23 + 15 + 27 + 7 + 15 + 27 + 7 + 15 + 12 + 19 + 13 + 19 + 13
	3 + 5 + 2 + 5 + 10 + 5 + 5 + 1 + 5 + 1 + 5 + 1 + 1 + 5 + 2 + 3 + 2 + 1
	1 + 10 + 10 + 25 + 5 + 1 + 1 + 1 + 2 + 2 + 3 + 1 + 5 + 1 + 1 + 1 + 1
	5 + 10 + 5 + 1 + 5 + 1 + 5 + 5 + 5 + 1 + 1 + 2 + 2 + 1 + 1 + 1 + 5
	16 + 30 + 14 + 28 + 16 + 30 + 27 + 22
	92 + 85 + 100 + 80 + 78 + 95
	88 + 85 + 95 + 80 + 90 + 90
	92 + 78 + 85 + 87
	85 + 65 + 80 + 81
	90 + 92 + 80 + 85
	92 + 90 + 88 + 90
	90 + 91 + 95 + 90 + 92 + 95
Mixed Numbers	$2\frac{3}{4} + 6 + 6\frac{1}{12}$
	$4\frac{1}{4} + 22\frac{1}{4} + 12\frac{1}{4}$
	$\frac{3}{4} + 2\frac{1}{2} + 8\frac{3}{4} + 4\frac{1}{4} + 5$
	$2 + 2 + 1\frac{1}{2} + 1\frac{1}{2}$
Decimals	.25 + .10 + .30
	.25 + .10 + .05
	.05 + .10 + .10
	3.75 + .45 + 1.70 + .50 + 1.40
	2.10 + 2.45 + 1.75 + 2.20 + 2.10
	.05 + .25 + .10 + .05
	.05 + .25 + .10 + .10
	.05 + .25 + .10 + .10
	.10 + .25 + .35 + .10 + .05 + .10 + .05
	.05 + .15 + .10 + .10
	42.80 + 32.00
	51.00 + 11.05
	10.00 + 2.00
	.15 + .20 + .40
	1.35 + 1.00 + .40
	11.00 + 3.12 + 2.40
	2.04 + 2.10 + 5.00
	108.61 + 74.80 + 100.00
	.13 + .16 + .09
	.20 + .75 + .15 + .20
	7.50 + .90 + 3.00 + 5.00
	26.40 + 20.75 + 2.00 + 18.40 + 6.75 + 4.00 + 10.00 + .60 + 2.55 + 3.15 + 1.11 + 8.50 + 1.30 + .25 + .45 + .40
	.11 + .11 + .03 + .29 + .22
	.07 + .33 + .12 + 2.00 + .17

CHART V (*Continued*)

Decimals (*Continued*)	5.50 + 2.75 + .35
	1.00 + 21.60
	8.99 + .75
	2.00 + .81
	3.00 + 3.60 + 5.67
	1.00 + .50 + .50
	.15 + .35 + .40 + .28
	.16 + .40 + .62 + .30
	4.00 + 9.60 + 18.75 + 14.00 + 10.00 + 2.50 + .40 + 1.25
	5.45 + 9.44 + 10.90 + 4.45 + 4.02 + 4.05 + 5.30 + 3.94 + 24.00
	1.00 + .40
	.20 + .40
	.10 + .20 + .05
	.05 + .25 + .05 + .10 + .10 + .05
	.15 + .15 + .30 + .35
	.10 + .10 + .20 + .50 + .30
	.30 + .50 + .10 + .40 + .10
	.20 + .20 + .50 + .20 + .40
	1.00 + .10 + .40 + .20
	.50 + .10 + .20 + .30 + .40
	152.2 + 151.9
	168.3 + 167.1
	236.4 + 16
	145.9 + 149.8
	92.2 + 89.4 + 95.6 + 96.4
	85 + 71.4 + 80.2 + 82.9
Other Types	3 yd. 2 in. + 2 in. + 3 in.

money transactions—decimals. The frequency of the 0 and 5 is noticeable here. The small frequency of fractions is again evident in Chart VI; also the simplicity of the few fractions is noteworthy. The computation of time lapsed is more frequent in this chart than in any other. One unit of work drew heavily on comparative dates and is responsible for this type of subtraction.

A study of multiplication in Grade 6. Chart VII presents a large group of multiplication computations made by pupils in Grade 6. Integers and decimals are most frequent. With integers there is a grouping of multiplicands of one- and two-place numbers and a second grouping of multiplicands of eight-or-more-place numbers. The larger multiplicands were supplied by units on the solar system.

The mixed numbers are fairly simple. The decimals are largely sums of money, although the use of 3.1416 as a multiplier in prob-

CHART VI

Problems in *Subtraction* Actually Done by Pupils of Grade 6

Integers	15,000,000 − 2,000,000	1933 − 1628
	15,000,000 − 2,000,000	100 − 46
	43 − 39	43 − 40
	43 − 26	43 − 36
	10 − 6	
Fractions	8/8 − 7/8	
Mixed Numbers	100 − 33⅓	3 − ½
	100 − 87½	
Decimals	10.00 − 7.80	2.50 − 2.20
	1.13 − .11	1.26 − .19
	.76 − .11	.88 − .09
	.45 − .30	.75 − .50
	.45 − .40	.50 − .25
	.50 − .25	1.00 − .35
	1.00 − .35	.50 − .35
	.50 − .35	.50 − .35
	.50 − .35	.50 − .35
	.50 − .35	.50 − .40
	.50 − .40	.50 − .40
	1.00 − .30	1.00 − .40
	1.00 − .40	1.00 − .45
	1.00 − .45	.50 − .30
	.20 − .15	.20 − .15
	.25 − .15	.50 − .15
	2.50 − 1.15	1.50 − 1.00
	3.65 − 2.75	5.00 − 1.64
	14.25 − 11.80	5.10 − 5.00
	16.52 − 15.08	15.08 − 1.08
	14.00 − 4.86	1.20 − .32
	283.41 − 62.05	2.00 − .76
	1.00 − .38	500.00 − 2.69
	20.00 − 8.50	3.25 − .39
	.60 − .50	4.50 − 3.75
	.40 − .30	22.60 − 12.25
	10.35 − 1.36	.50 − .34
	1.20 − .80	2.50 − 1.88
	3.00 − 2.70	71.55 − 60.50
	10.00 − 7.19	
Decimal	.15 − .13⅓	
Other Types	9/18/1634 − 8/14/1634	
	8/9/1612 − 7/20/1591	
	9/12/1657 − 11/15/1633	
	10/3/1637 − 9/18/1634	
	8/20/1643 − 7/20/1591	

CHART VII

PROBLEMS IN *Multiplication* ACTUALLY DONE BY PUPILS OF GRADE 6

Integers	5 × 50	9 × 30	10 × 30
	10 × 14	9 × 5	2 × 85
	9 × 14	9 × 11	4 × 19
	4 × 22	4 × 15	3 × 3
	2 × 5	3 × 10	23 × 23
	3 × 12	5 × 18	6 × 50
	17 × 30	12 × 60	4 × 100
	9 × 12	5 × 150	6 × 12
	2 × 50	4 × 8	5 × 10
	2 × 40	5 × 24	
	4 × 100	30 × 750,000,000	60 × 186,000
	640 × 523,802	640 × 523,802	
	11,160,000 × 60	669,600,000 × 24	
	16,074,400,000 × 360	186,000 × 60	
	11,160,000 × 60	669,600,000 × 24	
	16,074,400,000 × 360		
Mixed Numbers	$4 \times 12\frac{1}{2}$	$9 \times 12\frac{1}{2}$	$\frac{1}{2} \times 11$
	$2 \times \frac{3}{4}$	$4 \times \frac{1}{2}$	$5 \times \frac{27}{7}$
	$12\frac{7}{8} \times 17$	$1 \times \frac{1}{2}$	$2 \times 1\frac{1}{3}$
	$400 \times 7\frac{1}{4}$	$4 \times 1\frac{1}{4}$	$72 \times 1\frac{2}{3}$
	$1\frac{1}{4} \times 3\frac{3}{8}$	$22 \times 33\frac{1}{3}$	$2\frac{1}{2} \times 7{,}500$
	$3\frac{1}{2} \times 10$	$4\frac{1}{6} \times 45$	
Decimals	.10 × 1.13	.15 × 1.26	.15 × 76
	.10 × 88	.05 × 45	7 × .35
	.05 × 35	.10 × 22	.05 × 42
	2 × .05	2 × .05	2 × .05
	2 × .05	2 × .05	2 × .05
	1 × .25	1 × .25	1 × .35
	1 × .35	1 × .10	1 × .10
	1 × .05	1 × .05	1 × .05
	2 × .10	2 × .10	2 × .10
	2 × .10	3 × .05	3 × .05
	3 × .05	3 × .05	3 × .05
	4 × .05	4 × .05	4 × .05
	4 × .05	4 × .05	4 × .05
	4 × .05	4 × .05	2 × .25
	2 × .25	2 × .25	2 × .25
	2 × .15	2 × .15	2 × .15
	4 × .25	4 × .25	4 × .10
	4 × .10	4 × .10	4 × .10
	4 × .10	4 × .10	3 × .25
	1 × .15	100 × .0038	100 × .0089
	100 × .0091	100 × .0048	100 × .084
	100 × .036	100 × .039	100 × .10
	.46 × 2.50	.54 × 2.50	50 × 1.00
	30 × .0625	10 × .15	14 × .10
	1.15 × 1.56	.04 × 1.25	.10 × 1.50

CHART VII (*Continued*)

Decimals (*Continued*)	.15 × 100	6 × 39.4	22 × .50
	.12 × 26	8 × .30	9 × .12
	.12 × 17	7 × .30	10 × .50
	8 × 5.35	.40 × 66	.25 × 83
	5 × .40	16 × 1.15	5 × 1.35
	4 × 1.00	4 × 2.50	3 × .85
	7 × .45	3 × .37	10 × .85
	2 × .65	1 × .25	1 × .45
	1 × .60	6 × .40	.12 × 3.25
	40 × .25	40 × .25	40 × .05
	22 × .25	12 × .05	12 × .30
	10 × .10	30 × .72	7 × .18
	9 × 1.48		
	38,000,000 × 3.1416		
	67,000,000 × 3.1416		
	141,000,000 × 3.1416		
	96,000,000 × 3.1416		
	483,000,000 × 3.1416		
	886,000,000 × 3.1416		
	1,800,000,000 × 3.1416		
	2,800,000,000 × 3.1416		
Decimal Fractions	5 × .01¼		2½ × .25
	.50 × 7½		7½ × .60
Other Types		4 × 3 yds. 7 in.	

lems of constructing a planetarium introduces decimal computation in which no money is involved.

A study of division in Grade 6. Chart VIII shows the actual numbers used in division in Grade 6. In the list of integers two distinct groups of computations are found: one involves small numbers and the other group deals with very large numbers. As with multiplication in Grade 6, these large numbers are part of the number situation faced in a study of light and the solar system. Cancellation of ciphers is essential in the solution of these large, long-division numbers. In Chart VIII the lack of fractions is again obvious.

CONCLUSIONS AND RECOMMENDATIONS

Some characteristics of the problems surveyed. A summary of the outstanding characteristics as found in the survey is presented here:

CHART VIII

PROBLEMS IN *Division* ACTUALLY DONE BY PUPILS IN GRADE 6

Integers	18 ÷ 2	36 ÷ 2	26 ÷ 4
	6 ÷ 3	40 ÷ 5	50 ÷ 12
	15 ÷ 5	31 ÷ 7	900 ÷ 12
	6 ÷ 5	135 ÷ 7	120,000,000 ÷ 5
	40 ÷ 10	10 ÷ 4	3,000,000 ÷ 4
	13,000,000 ÷ 2,000,000		2,800,000,000 ÷ 36,000,000
	15,000,000 ÷ 335,233,280		9,500,000,000 ÷ 1,500,000,000
	183 ÷ 36		360,000,000 ÷ 1,500,000,000
	221,000 ÷ 300		135,000,000 ÷ 1,500,000,000
	736 ÷ 24		54,000,000 ÷ 1,500,000,000
	3,300 ÷ 870,000		11,000,000 ÷ 1,500,000,000
	7,700 ÷ 870,000		50,000 ÷ 1,500,000,000
	7,900 ÷ 870,000		186,000 ÷ 2,500
	73,000 ÷ 870,000		500 ÷ 30
	32,000 ÷ 870,000		720 ÷ 60
	34,000 ÷ 870,000		28 ÷ 36
	87,000 ÷ 870,000		75 ÷ 30
	67,200,000 ÷ 36,000,000		120 ÷ 200
	141,200,000 ÷ 36,000,000		13,800,000 ÷ 40,000,000
	96,000,000 ÷ 36,000,000		139,685,000 ÷ 57,255,000
	483,000,000 ÷ 36,000,000		350 ÷ 750
	886,000,000 ÷ 36,000,000		450,000,000 ÷ 750,000,000
	1,880,000,000 ÷ 36,000,000		750,000,000 ÷ 100,000,000
Mixed Numbers	3,610 ÷ 33⅛		7½ ÷ 2
Decimals	15,000,000,000 ÷ .03		295.7 ÷ 2
	.50 ÷ 1.50		304.1 ÷ 2
	335.4 ÷ 2		252.4 ÷ 9
	9.55 ÷ 72		51.00 ÷ .15
	10.00 ÷ 40		1.18 ÷ 3
	1.18 ÷ 4		9.74 ÷ 2
	71.55 ÷ 9		
Decimal Fractions	.10 ÷ 3½		

1. A very wide range of problem situations was found in both grades. (See page 92 and following.)

2. A higher percentage of noncomputational problems is found in Grade 3 than in Grade 6. (Table I.)

3. A higher percentage of computational problems is found in Grade 6 than in Grade 3. (Table I.)

4. Many more computations were necessary to solve the more complex problems of Grade 6. (Table II.)

5. Multiplication is the most frequently used fundamental process when both grades are combined. (Table II.)

6. Division is the least frequently used fundamental process when both grades are combined. (Table II.)

7. Nearly half of the computations for the two grades combined were with integers and of the remaining computations, decimals were next most frequently used. (Table III.)

8. Very few problems in either grade involve fractions, mixed numbers, or decimals, other than money. (Table III.)

9. A marked increase in complexity from integer to decimal computation in Grade 6 over Grade 3. (Table III.)

10. Integers are fairly evenly distributed among the fundamental processes in the two grades combined. (Table IV.)

11. Multiplication of decimals accounts for 46% of all computations in the two grades combined. (Table V.)

12. Division of decimals is seldom used. (Table V.)

13. Measuring, counting, and comparing are the most frequently found noncomputational problems. (Table VI.)

14. Graphing and scale drawing seem important in Grade 6. (Table VI.)

15. Grade 3 addition consist of all types of numbers, but in all instances, simple numbers. The only fractions used are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. (Chart I.)

16. Grade 3 subtraction is almost exclusively with integers. There are some mixed numbers. The fractions include $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. (Chart II.)

17. Grade 3 multiplication is confined largely to integers; the same is true of division. (Charts III and IV.)

18. Grade 6 addition is largely integers and decimals (money). Fractions are still simple—$\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{12}$. (Chart V.)

19. Grade 6 subtraction is mostly with decimals (money). (Chart VI.)

20. Grade 6 multiplication is largely with integers and decimals. (Chart VII.)

21. Grade 6 division is largely with integers with some large numbers. (Chart VIII.)

22. Problems containing decimals were largely of money transactions.

Some general comments. A few general comments seem to be in order at this point in the summary.

1. It was planned originally to check the findings of this survey with courses of study to discover inarticulations. However, the newer courses vary so greatly in the grade placement of number facts and processes that it seemed impractical to set up a standard against which to measure the findings of this survey. Consequently, each city or state may make its own comparison to see how well it squares with the findings of this study.

2. The present study is decidedly not a comprehensive survey. It takes into account only Grades 3 and 6. It does not run throughout the twelve months of the year. It lacks a record of many out-of-school number situations. It is not a fair geographic sampling of the pupils of the United States. For these and other reasons, this study should be considered only as a preliminary investigation.

3. The authors of this study feel that a national survey of situations in which children find a need for arithmetic is highly desirable. It is their belief that considerable inarticulation would be found between even the most modern arithmetic courses and the actual needs of pupils. Undoubtedly, there are many aspects of arithmetic now taught much too early—before the meaning and need have been experienced by the pupils. Also, many aspects considered essentially as parts of a carefully planned, sequential, and systematic program would be learned inductively, out of their logical order. Nevertheless they are learned and perhaps, in the long run, better learned as far as functioning when needed in subsequent behavior is concerned. A survey would focus attention on such inarticulations and unique learning characteristics of children. Such a study might contribute much to facilitate developing control over arithmetic facts and processes where meaning has been made clear through personal experience.

4. While the committee presenting this report believes the survey demonstrates a richness and vitality of arithmetic experiences in the activity program which may serve to give the pupils significant meaning and purpose, yet they point out again (see paragraphs under *A Third Viewpoint: Recognizing Two Goals*, pages 88 and 89) that "functional experiences of childhood are alone not adequate to develop arithmetic skills." The teacher should recognize these meaningful arithmetic experiences as readiness preceding the practice or drill necessary to fix the fact or process for the learner and

the teacher must provide sufficient periods of practice to assure mastery.

5. Further, the authors recognize that the present activity program does not assure a comprehensive orientation in arithmetic.* It is evident from this survey that each classroom found a very small number of experiences per week. In addition, there is a large element of chance operating in the selection of units of work in the present activity school. In the typical activity program, there is no check to assure that the total experiences of the six years of elementary school will introduce a child to most of the significant phases of comprehensive living in our contemporary world. If the activity program had in it some principle guaranteeing the development of a comprehensive understanding of most of the important social areas, then this committee would be willing to leave to the demands of such activities, the development of the necessary arithmetic fact and process, utilizing, of course, these meaningful situations as the occasions for drill sufficient to give the required degree of facility. They do not feel, however, that a survey of the opportunities found in the *present* activities program is a sufficient guide to the selection of arithmetic materials. Some guide or criterion in addition to opportunities-found-in-activities is essential, especially in an age where accurate quantitative thinking and computation is yearly becoming more basic to planning our collective enterprises. The most obvious source of this second criterion is the arithmetic needs of this collective society. When course-of-study makers can determine more clearly what constitutes full and comprehensive living in our society, then the teacher will have the important guide by which she may progressively select units of experience which will demand the quantitative thinking and skills

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* EDITOR'S NOTE: The 234 third grade problems represented the total number of problems that arose in the six third grades during a period of four months. This is an average of about 60 problems per month for the entire six grades, or an average of 10 problems per room per month. These 234 problems included both computational and noncomputational varieties. On page 101 it is stated that 56% of these problems were computational, which means that, on the average, only 5 computational problems per room per month were encountered.

Similarly, the 205 sixth grade problems represented the total number that were found in all sixth grades in four months. This makes an average of 8½ problems per room per month, of which 72% were computational and the rest noncomputational. This gives about 6 computational problems per month for each sixth grade room. It is this very small amount of number work encountered in each grade that leads the authors to the conclusion, stated above, that it is not possible to teach arithmetic solely through an activity program.

commensurate with modern living. The school has an obligation to society to see that all citizens develop sufficient arithmetic competency to carry intelligently their mutual responsibilities. Only as the units of work or activities in which the children engage are selected in terms of a total social experience today is there any assurance that the school can meet its obligations to society and the individual. In other words, the authors of this report contend that units of experience should be as comprehensive and as all-inclusive of social problems as is possible. Then and only then will a survey of arithmetic opportunities in such units give the major cues to selection and organization of classroom materials.

6. Until such time as the activities program is fundamentally reconstructed and a survey of these arithmetic opportunities made, a teacher will find it advantageous to approach the teaching of arithmetic through her own survey of the needs of her own pupils. If no opportunities are found for certain of the present courses-of-study requirements, she will probably do the best she can to build meaning before drill. But constantly, she will urge a revision of the curriculum in terms of a more socially comprehensive experience which will surely present the necessity to gain control over important arithmetic skills and processes.

Additional Unpublished Studies of Similar Nature

Fannie Dunn and Students. *A Suggestive List of Experiences Involving Number Such as Are Likely to be Found in Rural School Children's Environment.* Rural Education Department, Teachers College, Columbia University. Form 373. 6 pages, mimeographed.

Fannie Dunn and Students. *A Suggestive List of Experiences Involving Number in the Life of a Child in the Intermediate and Upper Grades of a Rural School.* Same as above. Form 375. 6 pages, mimeographed.

Ellen Todd Warner. *A Comparative Analysis of the Arithmetic of Two Third Grades.* Master's Thesis. Ohio State University. 1931.

Henry Harap and Charlotte Mapes. *Six Activity Units in Fractions.* Bulletin No. 33. December 1, 1933. Curriculum Laboratory, Western Reserve University. 19 pages, mimeographed.

ECONOMY IN TEACHING ARITHMETIC

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I. Introduction

Main purpose of this study. The underlying motive which prompted this article was the idea that mathematics need not and should not be a disliked subject. Children manifest a liking for it in the early grades as long as they understand it. However, along in the fourth, fifth, and sixth grades we begin pushing and crowding them in an effort to teach them all the fundamental skills, never stopping to ask ourselves whether or not the child is mentally mature enough to understand the many intricate processes. Skill after skill is taught before any meaning is attached to them. The child goes through a maze of mechanical operations, most of which he does not understand but merely memorizes momentarily. Then when he is ready for the junior or senior high school his taste for mathematics is perverted and he thinks that arithmetic consists of juggling figures and working hard examples. The teacher in her zeal continues to emphasize skills because the course of study calls for so many of them, and, not being able to put them all across, finds no time for the social aspect which is, after all, the more important. In an effort to show how economy can be effected in teaching the necessary skills so that some time can be provided for the social side of quantitative thinking this article is written.

Before taking up the experiment and the immediate reasons therefore, let us consider briefly the history of the relative status of common fractions and decimals.

Past and present trends in common fractions and decimals. The use of common fractions began in Egypt, as recorded in the Rhind Papyrus, about 1600 B.C. Decimals were first used in Europe about the year 1600 A.D., some three thousand years later.

It is interesting to note that the arithmetics written in England shortly after decimals were introduced gave more value to decimals than to common or vulgar fractions, as they were then called, and that later in this country the arithmetics stressed common fractions and are still stressing them to a much greater extent than decimals. A few specific cases will be cited.

In 1729 William Webster published in London his *Arithmetic in Epitome in Two Parts.* Part I was called "Vulgar Arithmetic" and Part II was called "Decimal Arithmetic." In Part I only twenty-three of 162 pages were devoted to vulgar fractions, whereas all of the fifty-five pages of Part II dealt with decimals and how to use them. In the preface to this book the author writes:

> The Second Part of this Treatice which contains the Doctrine of Decimal Fractions is a kind of Arithmetick peculiarly, as it were, adapted to the concerns of Gentlemen: I have therefore been more large than ordinary upon that Subject and have run over the General Rules again, to shew its particular use and application.

In the preface of another arithmetic, called *A New and Compendious System of Practical Arithmetick*, published by William Pardon in London in 1738, we find:

> But above all I would seriously recommend the Study and Practice of Decimals, whose superlative Excellence may be seen by the several Applications exhibited in this Book, and particularly the Contract Method* of Multiplication, which for its Ease and Expedition deserves to be universally practis'd by all Persons, where it can be applied. Another Reason for the Use of Decimals, is, that in all fix'd Cases you may turn your Divisor into a Multiplier, which will generally expedite the Work so much and render it so easy, that every one that tries it will soon find its Extraordinary Usefulness, especially now the methods of managing, repeating, and circulating Decimals are fully known.

After spending thirty-four pages explaining common fractions, followed by forty-six pages on decimals, the author adds a chapter of forty-one pages entitled, "Rules of Practice Wrought Diverse Ways Both Vulgarly and Decimally," in which chapter he shows the advantages of decimals over common fractions by giving the solution to numerous examples in multiplication of denominate numbers, mostly by finding costs of various quantities. One wonders why the author brought in decimals when using English money which is not on a decimal scale. An examination of the several examples solved, however, shows the actual labor saved. At the end of this double method of explanation he says:

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* By *contract method* is meant the method of rounding off.

> I have now so fully explained this Rule that I think there can remain no Difficulty, but what some or other of the foregoing Methods will easily master: and indeed I would recommend the Decimal Way to be used universally, both for its Ease and Certainty, not being incumbered with Variety of Denominations: for after the Parts of any Quantity whatever are found, either by Table or otherwise, then the Process is the same with plain simple Multiplication though the Product is applicable to any Species of Coin, Weight, Measure, etc.

Less than fifty years later, in 1786, in New York, Nicholas Pike published his arithmetic, *A New and Complete System of Arithmetic.* It contained 503 pages of which fifteen were devoted to vulgar fractions and only fifteen to decimals and federal money. The author says in the preface,

> The Federal Coin, being purely decimal, most naturally falls in after Decimal Fractions.

Two arithmetics issued in our country shortly after the adoption of our decimal system of coinage gave precedence to decimals. One in 1796 by Erastus Root treated the common fractions ½, ¼, and ¾ only. The other, by Chauncey Lee, published in 1797, gives this interesting comment:

> As the use of vulgar fractions may be advantageously superseded by that of decimals, they are viewed as an unnecessary branch of common school education and are therefore omitted from this compendium.

In 1821 Warren Colburn published his famous *Intellectual Arithmetic Upon the Inductive Method of Instruction* in this country. In it he gave seventy-six of the total 216 pages to common fractions and, strange to say, gave no discussion to decimals.

In 1850 an arithmetic by D. McCurdy appeared in Boston, in which decimals and federal money were taken up immediately after whole-number operations. Common fractions were introduced after percentage and its applications and just before involution and square root. This was the only textbook among those examined in which this arrangement was found.

In 1858, in New York, Robinson published his *Progressive Intellectual Arithmetic.* Of a total of 174 pages he devotes sixty to common fractions, followed by thirty pages more of problems involving such fractions as: "4/3 of 36 is 4/7 of how many times 2/7 of 42." The subject of decimals was left out entirely.

Robinson's Practical Arithmetic, published in New York nineteen years later, contains 360 pages of which twenty-six were given to common fractions and twelve to decimals.

For a clearer picture of the trend, the above dates and some later ones will be given in Table I.

TABLE I

TREND OF TEACHING COMMON FRACTIONS AND DECIMALS DURING THE LAST TWO CENTURIES, AS REVEALED IN LEADING ARITHMETIC TEXTS

	Date	Where Published	Total Pages	Common Fractions		Decimals		Ratio	
				Pages	%	Pages	%	Common Fractions	Decimals
1.	1665	(In Latin)*	383	13	3.4	20	5.2	1.0 :	1.5
2.	1729	England	217	23	10.7	55	25.0	1.0 :	2.3
3.	1738	England	397	34	8.6	46	11.6	1.0 :	1.3
4.	1751	England	401	23	5.7	56	14.0	1.0 :	2.4
5.	1786	United States	503	15	3.0	15	3.0	1.0 :	1.0
6.	1789	United States	220	15	6.8	9	4.1	1.7 :	1.0
7.	1821	United States	216	76	35.0	0	0.0	35.0 :	0
8.	1837	United States	228	17	7.5	11	4.8	1.5 :	1.0
9.	1847	United States	282	30	10.6	17	6.0	2.0 :	1.0
10.	1850	United States	258	17	6.6	13	5.0	1.3 :	1.0
11.	1858	United States	174	60	34.0	0	0	34.0 :	0
12.	1862	United States	239	25	10.5	9	3.8	2.8 :	1.0
13.	1873	Germany	336	66	19.6	2	0.6	33.0 :	1.0
14.	1877	United States	360	26	7.2	12	3.3	2.0 :	1.0
15.	1881	United States	359	38	10.6	26	7.2	1.5 :	1.0
16.	1885	United States	261	38	14.5	28	10.7	1.4 :	1.0
17.	1900	United States	338	24	7.1	9	2.6	3.0 :	1.0
18.	1902	United States	369	53	14.3	21	5.7	2.5 :	1.0
19.	1906	United States	788	195	25.0	37	25.0	5.0 :	1.0
20.	1913	United States	866	92	10.0	30	3.0	3.3 :	1.0
21.	1920	United States	806	70	8.7	33	4.3	2.0 :	1.0
22.	1925	United States	844	113	13.0	64	7.5	2.0 :	1.0
23.	1928	United States	1044	110	10.0	68	6.5	1.5 :	1.0
24.	1930	United States	1157	166	14.0	64	5.5	2.5 :	1.0
25.	1932	United States	1497	146	9.0	72	5.0	2.0 :	1.0
26.	1934	United States	1564	328	21.0	153	9.8	2.1 :	1.0

* Place not given.

Table I shows several trends. We notice that before 1786, as indicated by the first four items, the Common Fractions/Decimal ratio was about 1 to 2. After 1789 and up to the present time, leaving out the three extreme cases (7, 11, and 13), the ratio is reversed, being about 2:1. Why the arithmetics in this country after 1820 emphasized common fractions to such an extent and slighted decimals when the English and early American arithmetics had done the reverse is difficult to understand. This is all the more difficult to explain when we take into account the fact that English money is not built on a decimal basis and that United States money is. Can it be that Warren Colburn in his brilliant style of 1821 when discussing the method of Pestalozzi swept the arithmetic public before him and converted every American teacher to common fractions? Or did the tremendous sales of his book influence later writers of textbooks? Another explanation might be that the belief in the doctrine of mental discipline was so strong in those days among school people that common fractions were selected because they offered more meat for the mind. That Warren Colburn believed in mental discipline is revealed in the following quotation:

> Few exercises strengthen and mature the mind so much as arithmetical calculations, if the examples are made sufficiently simple to be understood by the pupil; because a regular, though simple, process of reasoning is requisite to perform them, and the results are attended with certainty.[1]

We note also from Table I that the increase in total number of pages has gone upward like a growth curve, from a modest 200 to 300 pages before 1900 to 1,564 pages in 1934. These totals are for books for the third to eighth grades of the elementary school. The increase in number of pages is also largely due to the increase in number of pages given over to common fractions and decimals, especially common fractions. Now, if we take into account that several editions of number primers for the second grade and even the first grade, not to mention the many arithmetic workbooks, have appeared recently, we wonder how long this expansion can keep up. Children, no doubt, can read better and more now than they could a hundred years ago, but if the other school departments also keep up an enlargement program there will come a time, perhaps it has already come, when we shall have to call a halt and take an inventory of our demands upon the child.

An examination of books on methods of teaching in foreign countries reveals further anomalies. In France, a book on method, *Leçons d'Arithmetique*, by Jules Tannery published in 1900 devotes but ten pages to common fractions and thirteen to decimals, out of a total of 509 pages. Another French book on method, *La Theorie Arithmetique*, by Le Moyne and Aymard, published in 1904, contains 434 pages, of which sixty pages are given to common fractions and thirty to decimals. Still later in 1912 appeared *Les Nombres Positifs*, a book on methodology by M. Stuyvaert. Of a total of 169 pages thirty are given to common fractions and none to decimals. One wonders again why there was such a growing emphasis on common fractions in a country in which the metric system is used.

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[1] Preface of first edition of *First Lessons in Intellectual Arithmetic*, by Warren Colburn. 1821.

In Germany, A. Gerlach's book on method in elementary arithmetic, *Lebenswoller Rechenunterricht*, appeared in 1920. This book criticizes the current practice among German arithmetic textbooks of giving so much space to the unused common fractions.

An interesting comment is found in a report by an Education Committee from a *Conference Report on Teaching Arithmetic in the London Elementary Schools* in 1911. This report reads as follows:

> It appears from the answers that very few teachers would, or do, postpone the systematic teaching of decimal fractions beyond the stage when a child possesses a clear conception of the nature of a vulgar fraction together with the ability to perform very simple operations with vulgar fractions. The majority both of men and women teachers agree that an introduction to the notation of decimal fractions can be usefully made even before the stage above mentioned has been reached; then would subsequently carry on the teaching of decimals and vulgar fractions simultaneously, carefully inculcating in the child a preference for working decimally unless there was a distinct gain in doing otherwise. A substantial minority point out that although training to work in decimal fractions is most valuable, yet of an equal value is the mental training derived from the manipulation of vulgar fractions in connection with problem work.

II. Reasons for Undertaking This Study

Inadequate results obtained by students at end of the sixth grade. Numerous investigations and experiments have shown that neither common fractions nor decimals are learned to any degree of mastery by students by the time they reach the end of the sixth grade. One study of this subject is the dissertation by Schorling, which gives the results of tests taken by 3,260 students entering the seventh grade in widely scattered cities in the United States. All of the questions involving common fractions and decimals and their results are given here in Table II.

TABLE II

PER CENT OF CORRECT RESPONSES TO QUESTIONS BY STUDENTS ENTERING GRADE 7 (ADAPTED FROM SCHORLING)[2]

In Common Fractions

	%
1. What must you do to find $\frac{2}{3}$ of $\frac{5}{6}$? Do you add? If not, what must you do?	54.9
2. Draw a circle around the part of the fraction $\frac{4}{5}$ which is called the denominator.	41.2
3. Can we multiply both terms of a fraction by the same number without changing the value of the fraction?	23.4
4. Can we subtract the same number from both terms of a fraction without changing the value of the fraction?	22.6
5. Can we divide both terms of a fraction by the same number without changing the value of the fraction?	5.2
6. Can we add the same number to both terms of a fraction without changing the value of the fraction?	0.4

In Decimals

1. Write .25 as a common fraction.	81.3
2. Write in figures. Fifty-nine and three hundredths. Write here.	76.1
3. Write $\frac{3}{4}$ as a decimal.	74.8
4. Write $\frac{1}{8}$ as a decimal.	71.4
5. Write $\frac{1}{3}$ as a decimal.	66.6
6. Does 1.2 × .5 equal 6.0 or .60 or .060 or 60? Draw a circle around the right number.	63.7
7. Write $\frac{1}{16}$ as a decimal.	55.4
8. Moving the decimal point one place to the right in a number .... the number by 10.	49.5
9. Write .125 as a common fraction.	47.5
10. Moving the decimal point one place to the left in a number .... the number by 10.	42.1
11. Write $\frac{1}{12}$ as a decimal.	35.9

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[2] Schorling, Raleigh, *A Tentative List of Objectives in the Teaching of Junior High School Mathematics* pp. 25-77.

The above questions are mainly on the meaning of common fractions and decimals.

A test on manipulations in the two operations was given by the writer to a 6A class of 43 students at the end of the year.

The tests and results follow:

Common Fractions	Per Cent Correct
1. Add $3\frac{5}{8}$ to $4\frac{5}{6}$	68
2. Subtract $8\frac{1}{2}$ from $10\frac{1}{4}$	75
3. Multiply $3\frac{1}{8}$ by $\frac{4}{5}$	33
4. Divide: $4\frac{1}{4} + 5$	23
5. $\frac{2}{5} + \frac{3}{4} = ?$	46

Decimals	Per Cent Correct
1. Add 3.95 and 4.5	59
2. Subtract 6.5 from 8.25	56
3. Multiply 1.62 by .32	26
4. Divide: 7.32 ÷ 4	50
5. Divide: $.21\overline{)44.1}$	32

Let any teacher give these tests to an entering seventh grade not previously coached and note the results.

The results obtained in common fractions and decimals at the end of the sixth grade do not warrant the time and energy put forth by both students and teacher in the several years of teaching and learning of these two topics. Something is wrong somewhere.

The skills required in addition and subtraction of common fractions too difficult for fourth and fifth grade pupils. That this has been the cause of the poor results shown in the preceding section has been believed by some and doubted by others. This need no longer be doubted. The Committee of Seven by their long and comprehensive experiment has shown that addition and subtraction of common fractions are too difficult for the fifth grade, and that they are much more difficult than addition and subtraction of decimals. The committee is now in the tenth year of its experiment which represents fifteen different states and several hundred cities.[3]

The minimum mental age and minimum grade at which a topic in arithmetic should be taught is reported in their grade placement table. Table III here gives that part of their table which deals with the topics relating to common fractions and decimals.

TABLE III

MINIMUM MENTAL AGE AND MINIMUM GRADE PLACEMENT OF CERTAIN TOPICS IN ARITHMETIC

Topic	Minimum Mental Age	Minimum Grade
Addition and subtraction of decimals	10–11	5
Memorization of fractional and decimal equivalents	11–6	5
Multiplication of fractions	12–3	6
Division of fractions	12–3	6
Division of decimals	13–0	7
Addition and subtraction of fractions and mixed numbers with unlike denominators (involving borrowing)	13–10	7

The 1931 report of the Committee of Seven says:[4]

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[3] Storm, H. C. For personnel of the committee and the details of its method see "Grade Placement—A Summary of the Findings of the Committee of Seven," *The Illinois Teacher*, December 1931.

[4] *Ibid.*

The committee is convinced that teachers are very generally trying to do the impossible in arithmetic in forcing upon the child computations that are not at all fitted to his growth. Our courses of study as well as our textbooks are too much influenced by tradition and are not nearly enough the result of scientific investigation. This blind following of tradition causes much heartache on the part of both teachers and pupils and engenders hatred of certain phases of arithmetic, whereas the whole field of arithmetic should be a constant source of joy to teacher and pupil alike.

The number of skills in common fractions and decimals requiring mastery too many for the time allotted to them. The many recent analyses of the various processes in arithmetic made by Brueckner, Knight, Morton, Osburn, Wells, and others have had some fruitful results. It may be true that some of these authors have gone into such detail in unit skills and types that the woods cannot be seen for the trees but they have shown us what there is to teach and to learn in the various processes. They have revealed to us the relative numbers of skills in the various operations, which alone is worth while, not to mention the valuable help in diagnostic testing and remedial teaching that these analyses have given.

In an effort to obtain a more definite measure of the teaching time of these skills in common fractions and decimals, the writer has made a less detailed analysis into major skills. A major skill consists of one or more unit skills that go to make up one complete act. An illustration will make this clear. Carrying in addition is called a major skill by the writer although it is made up of three unit skills, according to Brueckner, for example: writing down the right-hand figure only, carrying a number in mind, and adding the carried number to the first number in the next column. Since these are parts of one major ability, that of carrying, and since the act of carrying cannot be completely performed without all of the parts they are classified as one major skill. This classification into major skills is better for teaching purposes, albeit the unit skills are better for diagnostic purposes.

According to the above method of analysis common fractions and decimals were analysed by the writer as shown in Tables IV and V which follow:

TABLE IV

The Major Skills in Common Fractions

	Illustrations of Skill		
I. Meaning:			
1. Fractions of a whole.	$\frac{1}{4}$ of 1	$\frac{3}{4}$ of 1	
2. Fractions of a group.	$\frac{1}{3}$ of 12	$\frac{2}{3}$ of 12	
II. Reduction:			
1. To lower terms and higher terms.	$\frac{4}{8}=\frac{1}{2}$	$\frac{1}{2}=\frac{3}{6}$	
2. From improper fraction to mixed number and the reverse.	$\frac{9}{8}=1\frac{1}{8}$	$2\frac{1}{4}=\frac{9}{4}$	
III. Addition:			
1. Fractions and mixed numbers whose denominators are alike.	$\frac{3}{8}+\frac{4}{8}$	$1\frac{2}{3}+3\frac{2}{3}$	
2. Fractions and mixed numbers whose denominators are unlike, one denominator being a multiple of the other.	$\frac{1}{2}+\frac{3}{4}$	$4\frac{5}{8}+2\frac{1}{4}$	
3. Fractions and mixed numbers whose denominators are prime to each other.	$\frac{1}{3}+\frac{1}{4}$	$6\frac{1}{2}+2\frac{1}{3}$	
4. Fractions and mixed numbers whose denominators contain a common factor, one denominator not a multiple of the other.	$\frac{3}{4}+\frac{1}{6}$	$1\frac{7}{8}+5\frac{1}{6}$	
IV. Subtraction:			
1. Fractions from fractions and mixed numbers from mixed numbers (minuend fraction greater than subtrahend fraction).	$\frac{5}{6}-\frac{3}{4}$	$12\frac{3}{4}-8\frac{1}{2}$	
2. Whole numbers from mixed numbers and mixed numbers from whole numbers.	$14\frac{7}{8}$ 11	16 $7\frac{5}{8}$	
3. Mixed numbers from mixed numbers (minuend fraction smaller than subtrahend fraction).	$34\frac{1}{2}$ $15\frac{3}{4}$		
4. Fractions from mixed numbers (minuend fraction smaller than subtrahend fraction).	$6\frac{2}{3}$ $\frac{5}{6}$		
V. Multiplication:			
1. Fractions or whole numbers by fractions or whole numbers.	$\frac{1}{2}\times\frac{3}{4}$	$3\times\frac{3}{4}$	$\frac{1}{3}\times 6$
2. Mixed or whole numbers by mixed or whole numbers.	$2\frac{1}{2}\times 3\frac{1}{4}$	$3\times 4\frac{1}{2}$	$6\frac{1}{2}\times 3$
3. Large mixed or whole numbers by large mixed or whole numbers.	$16\frac{2}{3}$ 12	24 $13\frac{3}{4}$	$184\frac{1}{2}$ $36\frac{3}{4}$
VI. Division:			
1. Whole numbers or fractions by whole numbers or fractions.	$2\div\frac{1}{3}$	$\frac{1}{3}\div 2$	$\frac{1}{4}\div\frac{1}{3}$
2. Mixed or whole numbers by mixed or whole numbers.	$2\frac{1}{2}\div 1\frac{1}{4}$	$3\frac{1}{4}\div 4$	$8\div 6\frac{1}{2}$
3. Mixed numbers or fractions by mixed numbers or fractions.	$7\frac{1}{2}\div\frac{3}{4}$	$\frac{7}{8}\div 3\frac{1}{2}$	

Total number of major skills in common fractions 18

TABLE V

THE MAJOR SKILLS IN DECIMALS

I. Writing and meaning:	
1. Writing and reading.	.606 600.006
2. Meaning.	.1 is larger than .09
II. Addition:	3.8 + .37
III. Subtraction:	14. − .69
IV. Multiplication:	$\begin{array}{r}12.25\\ 3.6\\ \hline\end{array}$
V. Division:	
1. Decimals by integers.	$6\overline{)\,.72}$
2. Decimals by decimals.	$.8\overline{)\,.56}$
3. Integers by decimals.	$.17\overline{)\,93.}$
4. Zero difficulties.	$.12\overline{)\,.006}$
5. Integers by integers (with remainders).	$63\overline{)\,79}$
VI. Changing common fractions to decimals.	$^5/_8 = .625$

Total number of major skills in decimals	11
Total skills in common fractions and decimals	29

Brueckner in his diagnostic studies in arithmetic lists the following types and varieties of errors in common fractions and decimals:

In Common Fractions:

Number of types from an *a priori* analysis:[5]

In addition	40
In subtraction	45
In multiplication	45
In division	40
Total	170

Number of varieties of errors found in an analysis of 21,065 errors from 83,800 examples by 600 pupils in grades 5A, 6B, and 6A:[6]

In addition	32
In subtraction	44
In multiplication	19
In division	20
Total	115

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[5] Brueckner, L. J., *Diagnostic and Remedial Teaching in Arithmetic*, pp. 177-194.

[6] *Ibid.*, pp. 200-206.

In Decimals:

Number of types from a made-up diagnostic test:[7]

In addition	12
In subtraction	17
In multiplication	32
In division	30
Total	91

Number of varieties of errors found in an analysis of 8,785 errors from 15,288 examples by 168 pupils in Grade 7.[8]

In addition	15
In subtraction	14
In multiplication	26
In division	26
Total	81

Teachers who are opposed to analyses will discount these summaries and say it is an impossible task to teach so many different types as separate skills to be learned. Be that as it may, if we do not agree with the *a priori* analyses in the first and third summaries above, we must agree with the second and fourth summaries which are taken from empirical data in actual tests taken and that there were that many different kinds of errors found.

The reason for enumerating these types here is that the author is anxious to show that, in the second summary, the one based on analysis of actual errors in common fractions, the number of types of errors in addition and subtraction together are about twice the number in multiplication and division together, the ratio being 76 to 39. In the corresponding summary in decimals (see the fourth summary) the ratio of the number of types of errors in addition and subtraction to the total number in multiplication and division is just about reversed, being 29 to 52. Surely this is significant. This significance will be more fully treated later.

Furthermore, in connection with this fourth summary the following should be taken into consideration. In Brueckner's original tables from which this summary is derived there are three parts, as follows:

[illegible]*d.*, pp. 220-223.

[illegible]*d.*, p. 228.

1. Difficulties basic to any addition (subtraction, multiplication, or division).
2. Difficulties peculiar to the decimal situation.
3. Other difficulties (omissions, incompleted work, miscopying, etc.)

Counting the number of the varieties of the errors in each operation due to the decimal situation alone, the figures in summary 4 are reduced as follows:

In addition, from 15 to 5.
In subtraction, from 14 to 3.
In multiplication, from 26 to 10.
In division, from 26 to 10.

This reduces the total number of varieties of errors due to the decimal situation alone from 81 to 28.

Osburn, in an error study not so detailed as Brueckner's, lists the following varieties in common fractions and decimals:[9]

	Common Fractions	Decimals
In addition	6	1
In subtraction	5	1
In multiplication	9	1
In division	3	2
Total	23	5

In a later study Osburn proposed an estimation of relative difficulty of fractions based upon job analysis, that is, the number of steps actually used in different examples.[10] To illustrate: in the example, $7\frac{1}{12} - 4\frac{1}{3}$, he analyses and labels the steps as follows:

$$\begin{array}{r} 7\frac{1}{12} = 6\frac{13}{12} \\ \underline{4\frac{1}{3} = 4\frac{4}{12}} \\ 2\frac{9}{12} = 2\frac{3}{4} \end{array}$$

12 ÷ 3	1 × 4	12 + 1	7 − 1	13 − 4	9 ÷ 3	12 ÷ 3	6 − 4
D	M	A	S	S	D	D	S

There are 8 steps symbolized by the letters above. In the example, $4\frac{1}{2} - 2\frac{2}{3}$, there are 10 steps, which are, in summary, as follows: M, D, M, D, M, A, A, S, S, A. In the example, 3½ × ⅗, there are 5 steps: M, A, M, M, D. In the example, 6 ÷ ¼, there are only two steps, I and M (I signifies invert). In the example, 7½ − 5⅓, there are 10 steps: M, A, M, A, I, M, M, D, M, S.

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[9] Osburn, W. J., *Corrective Arithmetic*, pp. 47-55. Houghton Mifflin Company, 1924.

[10] Osburn, W. J., *Corrective Arithmetic*, Vol. II, pp. 39-46, 1929.

With so many steps involved we should not be surprised if we find that children have difficulty with common fractions.

Osburn made no such job analysis of decimals. Perhaps he thought they were not sufficiently difficult to warrant the effort. I am sure, however, that the division of decimals would embrace a number of steps, but the majority of these belong to ordinary division of whole numbers.

Summarizing and collecting the data set forth so far in this section, we find the number of skills and varieties of errors to be as follows:

	Common Fractions	Decimals
In writer's (major skills)	18	11
In Brueckner's (errors)	115	28
In Osburn's (errors)	23	5
Average	52	14⅔

If we differentiate between addition and subtraction on the one hand and multiplication and division on the other, we find the following figures:

	Common Fractions		Decimals	
	Addition and Subtraction	Multiplication and Division	Addition and Subtraction	Multiplication and Division
In writer's (skills)	8	6	2	6
In Brueckner's (errors)	76	39	8	20
In Osburn's (errors)	11	12	2	3
Average	31⅔	19	4	9⅔

It cannot be said that the writer's position on common fractions is extreme or unwarranted, for his figures are more modest than either those of Brueckner or Osburn. Nor did either of these men have any reason to be biased as they were interested solely in listing errors in both common fractions and decimals.

That the addition and subtraction skills in decimals are much simpler than the multiplication and division skills is shown in another summary—Table E on page 227 of Brueckner's book where we note that the median error in each of the operations is as follows:

In addition	8.0
In subtraction	5.0
In multiplication	14.5
In division	34.0

The above figures are based on errors actually made by 168 seventh grade pupils and show that the number of errors in multiplication and division together are nearly four times those of addition and subtraction combined.

These figures are not necessary in order to convince the teachers of arithmetic in the fifth and sixth grades. They know too well that there are too many items in common fractions and decimals both to be taught to a point of mastery in the time allotted to them in the course of study. They also know that common fractions are more difficult than decimals and that addition and subtraction of decimals are distinctly more simple and easier to teach than the corresponding operations in common fractions. The tables and figures are for the teachers in the secondary school who have never taught in the elementary school, and most of them have not.

As the writer has come in contact with these grades in direct teaching, in supervising in arithmetic, and in training teachers for teaching arithmetic in these grades, the problem is very definite to him and now seems acute. Something should be done to solve this problem and the writer is optimistic enough to believe that something can be done to relieve the situation.

If addition and subtraction of decimals in which there are so few skills to be learned could replace addition and subtraction of common fractions, in which most of the skills are, 50 per cent of the amount of work done in both could be cut down. To test out the feasibility of this is the purpose of the experiment reported later in this chapter.

Relative ease in learning and using common fractions and decimals in addition and subtraction. Compared with decimals, common fractions require many extra manipulations. For instance, in common fractions we have to learn how to reduce mixed numbers to improper fractions. There are no such reductions in decimals. We have to know how to reduce improper fractions to mixed numbers. There is no such reduction in decimals. We have to reduce to lower terms and higher terms in common fractions. In decimals we merely drop and annex zeros. In changing to a common denominator in common fractions, a fraction such as $\frac{1}{2}$ may be changed to a variety of different fractions, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, etc., depending upon the fractions with which it is to be combined, whereas in decimals the equivalent to $\frac{1}{2}$ is always .5. The concept of decimals is simply an extension of the concept of our whole-number system. The relative place value in our system of whole numbers obtains throughout the decimal system and also between the decimal system and the system of whole numbers. Thus the concepts of the two systems are mutually reinforcing.

An examination of the following examples will reveal the enormous amount of time saved when performing addition and subtraction of common fractions by means of their decimal equivalents.

Let it be required to add $3\frac{3}{4}$ to $2\frac{1}{2}$.

By Old Method	By Decimal Equivalents
$3\frac{3}{4} = 3\frac{3}{4}$	3.75
$2\frac{1}{2} = 2\frac{2}{4}$	2.5
$5\frac{5}{4} = 6\frac{1}{4}$	6.25

Let us work the example in subtraction, $34\frac{1}{2} - 15\frac{3}{4}$, by both methods:

By Old Method	By Decimal Method
$34\frac{1}{2} = 34\frac{2}{4} = 33\frac{6}{4}$	34.5
$15\frac{3}{4} = 15\frac{3}{4} = 15\frac{3}{4}$	15.75
$18\frac{3}{4}$	18.75

In the decimal solution of this example only one new skill is required, that of knowing the decimal equivalents. The remaining work is the same as in whole-number subtraction. In the old method at least five skills are required: reducing to a common denominator, borrowing from the whole number, changing a whole number to a fraction, changing a mixed number to a fraction, and subtracting numerators only.

These comparisons by no means tell the whole story. Although there is less writing to do in the final solution by the decimal method, the number of skills requiring mastery in learning are much fewer in decimals than in common fractions. Aside from this there is the question of errors. The chances for making errors are greatly reduced in the decimal method and this means that the remedial teaching is simplified. Also the errors are more easily detected in the decimal method, as will be shown later.

The use of decimals in multiplication and in division of common fractions is not taken up in this study or recommended for two reasons: first, multiplication and division of common fractions are much simpler than addition and subtraction, as has been shown, and, second, when multiplication and division of decimals are used the question of rounding off, or of approximations, becomes too complex for the fifth grade student. The rounding off from 2 to 1 and from 3 to 2 decimal places is simple and can and should be learned by fifth grade students as it is very useful in carrying out long division and for work in changing common fractions to decimals. This is all that is required in addition and subtraction of fractions by the decimal method.

The increasing importance and use of decimals. Another reason for justifying this procedure in present-day arithmetic would seem to be that decimals have become more and more important since the advent of the automobile. Decimals are rapidly replacing common fractions because in very accurate and minute measurements it is necessary to go beyond $1/64$ of an inch. The $1/128$, $1/256$, and the $1/1024$ inch are too unwieldy and laborious to handle, whereas the .01, .001, and .0001 inch are not. Aside from this, decimals offer a decided advantage when it comes to computing with them. This is why mechanics has decimalized the inch, engineering has decimalized the foot, and aviation has decimalized the mile. It is of interest to note in this connection that Henry Ford has ordered all fractions appearing on blue prints throughout his plant to be written as decimals.

The great need for economy and efficiency. At the present time when so many subjects are crowding in upon our elementary school curriculum and less time and importance is given to arithmetic than formerly, any attempt at an economy program in teaching arithmetic should be welcomed as a desirable move. Especially is this true if the program is to be more efficient.

The trend in recent courses of study in arithmetic is to ease up on the requirements in the lower grades. All of the addition and subtraction facts are no longer required to be completed until the third grade. The multiplication and division facts are not completed before the fourth grade. All of the other higher skills in all of the four fundamentals have to be taught somewhere in the elementary grades. The difficult long division which used to be taught in the fourth grade is now moved up to the fifth grade, and rightly so. Then comes common fractions which is begun in the fifth grade. The fundamental processes are not yet fixed, for the fifth grade child, a ten-year-old boy or girl, is very immature. One would not ask his own ten-year-old child to add up his grocery bill or check his bank stubs, yet at school the child of the same age is supposed to do much more difficult work. The result of all this is congestion in arithmetic learning in the fifth grade which causes confusion. The following year the conscientious teacher, aware of the fact that her students should know common fractions but that they do not, spends much time reteaching them to the neglect of the more important decimals. The student reaches the end of the sixth grade, when, according to the course of study, these skills should be known, with a hazy and imperfect notion of what fractions and decimals really are.

The writer, after visiting extensively and actually teaching all of these skills to classes in the fifth and sixth grades for many years, is firmly convinced that not more than the upper 20 per cent of the average fifth and sixth grades can be taught to master the skills necessary for fractions and decimals in the time alloted for them. It is pathetic for the conscientious teacher to slavishly follow the course of study in trying to teach the many skills and processes in long division and common fractions. In most places the children are willing enough and work hard to please the teacher, but to find difficulty after difficulty coming up with new skills to be learned before the old ones are mastered causes the child to be discouraged. Discouragement grows into disinterestedness and disinterestedness into dislike. Why do many of our high school girls and boys dislike mathematics? The answer is they were not taught correctly in the grades. Why do so many of our educationists knock mathematics? The answer is the same. They were not properly taught in the grades.

There has been much experimentation and some reform in secondary school mathematics but very little reform has taken place in the elementary field, aside from leaving out some unnecessary topics. In order to instill a love for mathematics which carries over into the high school, we must see that the child's taste is not perverted in the grades.

III. The Experiment

Object of the experiment. In the light of the foregoing reasons the purpose of this experiment was to find out if economy in time could be effected by teaching addition and subtraction of fractions by means of decimal equivalents and at the same time to find out if they could be taught more efficiently by this method.

Method. Before this experiment was begun permission was obtained from the assistant superintendent of schools in charge of experimentation. Further permission had to be obtained from district superintendents and principals. It was thought best to obtain this permission so that the experiment could go on unhampered. Conferences with these superintendents and principals were arranged and after some explanation as to the purport of the experiment consent was freely given. Fifth grade teachers who were in sympathy with the experiment were selected.

Since some of the teaching in this unit was rather new, suggestions setting forth the purpose and the prerequisite for the unit on teaching addition and subtraction by decimal equivalents were given to the teachers. A plan for presenting the unit was also given to the teachers for them to follow or to be guided by. In this presentation there are several fraction combinations that may not be found in real life, such as those on pages 142 and 143 in examples 11 to 28. The only reason that these were included was because they are found in the arithmetics of to-day and students are forced to work them by teachers who follow the book and most teachers follow the book.

Until we are sure which fraction combinations are and which ones are not used in the world around us, the decimal solution offers no additional difficulty in addition or subtraction, be the combination real or artificial, when the denominators are prime to each other. For example, fourths plus fifths and fifths plus eighths are much easier to add or subtract decimally than thirds plus fourths, although the former are said to be artificial combinations and the latter is real. In common fractions the relative facility is reversed.

The suggestions and presentation are reproduced in full herewith.

Miss L. began the work with her 5B class on March 13, 1933, after having spent about five weeks on the meaning of common fractions taught in the usual way. A careful check of the time was kept and recorded for future reference. Three weeks were spent on the meaning of decimals, three weeks on addition and subtraction of decimals, and three weeks on the new unit of work on decimal equivalents and addition and subtraction of common fractions by decimal equivalents. Periods of from 30 to 40 minutes each day were used.

Suggestions to Teachers for Teaching the Unit on Addition and Subtraction of Common Fractions by Decimal Equivalents

This unit presupposes that the student has had two units on the meaning of common fractions (including $\frac{3}{4}$ of 128, and the like, and addition of fractions of like denominators) and one unit on the meaning of decimals and one on the addition and subtraction of decimals. He is then ready to take up the 15 easy decimal equivalents of the frequently used common fractions by finding them as parts of a dollar, as shown later.

After these decimal equivalents have been learned the student can then add or subtract any of these common fractions or mixed numbers by simply writing their decimal equivalents and adding or subtracting as in whole numbers.

The great economy thus effected is at once seen, for the great variety of common denominators is hereby avoided. For example, $\frac{3}{4}$ when added will always be written as .75, whereas by the old method it has to be changed sometimes to $\frac{6}{8}$, sometimes to $\frac{9}{12}$, $\frac{12}{16}$, or $\frac{15}{20}$, depending upon the fraction to which it is added. Whereas 8 units are ordinarily required to teach addition and subtraction of common fractions, because there are that many varieties of least-common-denominator and mixed-number combinations, by the decimal method, but one unit is required. The actual simplicity will be seen in the presentation given below.

It is not intended that the student should convert his decimal answer back to a common fraction in the cases where the answers are not known decimal equivalents. In the majority of cases the answers will not be known equivalents. The answers, however, should have just as much or more meaning if thought of in terms of tenths or hundredths. For example, in Example 11 of the presentation .15 mile will give a meaning to the child of the fraction of a mile that the world uses to-day. We hear and read of .15 mile but not very often of $\frac{3}{20}$ mile. The child knows the meaning of .15 of a dollar, and .15 of a mile is the same fraction.

From his knowledge of dollars and cents the student knows the meaning of hundredths much better than thousandths. He is therefore asked to round his 3-place answers to the nearest hundredth to give them meaning. In Example 23 of the presentation .97 means more to a fifth grade child than .967 means. The fact that his monthly grades are given in terms of hundredths would help to give meaning to hundredths here also. He adds and subtracts the decimals to three places for the sake of accuracy but to give the answer meaning he should be able to read it to the nearest hundredth.

Presentation

I. *Learning how to find the 15 decimal equivalents*: You have studied the meaning of common fractions and decimals in previous units. You will study in this unit the relation between some of the common fractions and decimals. You already know the relation between $\frac{1}{2}$ and .50.

1. What is the difference between $\frac{1}{2}$ and .50? We say that \$.50 = $\frac{1}{2}$ of a dollar, so .50 of anything = what part of it?

2. You know that $\frac{1}{2}$ and $\frac{1}{2}$ make a whole. Add .50 and .50 to see if it makes 1 whole.

We want to learn how to write the simple common fractions as decimals so that we can add them easily.

3. What is $\frac{1}{4}$ of a dollar? Write it decimally.

4. What does $\frac{3}{4}$ equal decimally?

5. How much does $\frac{1}{4}$ and $\frac{3}{4}$ make? Add their decimal equivalents to see if they make 1.

6. Next we have the fifths: $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$. These can also be written as decimals for you know that $\frac{1}{5}$ of a dollar is \$.20 and $\frac{2}{5}$ of a dollar is \$.40. What is $\frac{3}{5}$ of a dollar?

7. What is $\frac{4}{5}$ of a dollar written as a decimal?

8. Write now the decimal equivalents of $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$.

9. What is $\frac{1}{5} + \frac{4}{5}$? Add their decimal equivalents.

10. How much is $\frac{2}{5} + \frac{3}{5}$? Add their decimal equivalents.

11. How would you find the decimal that is equivalent to $\frac{1}{8}$? In finding $\frac{1}{8}$ of a dollar divide \$1.00 by 8 and carry your answer to three places, thus:

$$\begin{array}{r} .125 \\ 8\overline{)\,1.000} \end{array}$$

12. Find the decimal for $\frac{3}{8}$ by multiplying the answer above by 3.

13. Find the decimal for $\frac{5}{8}$.

14. Find the decimal equivalent of $\frac{7}{8}$.

15. Add $\frac{1}{8}$ and $\frac{7}{8}$. Add their decimal equivalents.

16. Add $\frac{3}{8}$ and $\frac{5}{8}$. Add their decimal equivalents.

17. There are only four more common fractions whose decimal equivalents you should know. They are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, and $\frac{5}{6}$. Find what $\frac{1}{3}$ of a dollar is decimally, just as you found $\frac{1}{8}$ of a dollar.

18. Find the equivalent of $\frac{2}{3}$ by finding $\frac{2}{3}$ of a dollar. Do this just as you find $\frac{2}{3}$ of any number, for example,

$$\begin{array}{r} .666\frac{2}{3} \\ 3\overline{)\,2.000} \end{array}$$

As $6\frac{2}{3}$ is nearer 7 than 6, we shall write the equivalent as .667.

19. Find the decimal equivalent of $\frac{1}{6}$ and $\frac{5}{6}$ in the same way. The last four decimal equivalents are usually written thus:

$\frac{1}{3} = .333$ $\frac{2}{3} = .667$ $\frac{1}{6} = .167$ $\frac{5}{6} = .833$

20. Add $\frac{1}{3}$ and $\frac{2}{3}$ decimally. What should you get?

21. Add $\frac{1}{6}$ and $\frac{5}{6}$ decimally.

We have now learned the 15 decimal equivalents to the most commonly used fractions. You should write them in your notebooks in four groups, ranging from easy to difficult, as follows, and learn them.

$\frac{1}{2} =$
$\frac{1}{4} =$
$\frac{3}{4} =$

$\frac{1}{5} =$
$\frac{2}{5} =$
$\frac{3}{5} =$
$\frac{4}{5} =$

$\frac{1}{8} =$
$\frac{3}{8} =$
$\frac{5}{8} =$
$\frac{7}{8} =$

$\frac{1}{3} =$
$\frac{2}{3} =$
$\frac{1}{6} =$
$\frac{5}{6} =$

The following four equivalents are sometimes used. You need not learn them unless you want to. They can be referred to if you have fractions like them to add or to subtract.

$\frac{1}{12} = .083$ $\frac{5}{12} = .417$ $\frac{7}{12} = .583$ $\frac{11}{12} = .917$

Note to the teacher: Whether the twelfths should be memorized or merely used as references is left to the instructor. In the practice exercises of this unit they are left out as a minimum requirement.

II. *Learning how to add by means of decimal equivalents:*

1. In making fudge Mary used $1\frac{1}{2}$ cups of brown sugar and $\frac{3}{4}$ of a cup of granulated sugar. How many cups of sugar did she use?

$$\begin{array}{r} 1.5 \\ \underline{.75} \\ 2.25 \end{array}$$

By writing the $\frac{1}{2}$ and $\frac{3}{4}$ as decimals, you can add them easily. By adding you get 2.25 or $2\frac{1}{4}$ cups.

Add the following as you did Example I:

2. $\begin{array}{r} 3\frac{1}{2} \\ \underline{2\frac{1}{4}} \end{array}$ 3. $\begin{array}{r} 2\frac{2}{3} \\ \underline{1\frac{1}{2}} \end{array}$ 4. $\begin{array}{r} 3\frac{5}{6} \\ \underline{4\frac{1}{2}} \end{array}$ 5. $\begin{array}{r} 2\frac{1}{4} \\ \underline{\frac{3}{8}} \end{array}$ 6. $\begin{array}{r} 6\frac{1}{2} \\ \underline{8\frac{7}{8}} \end{array}$ 7. $\begin{array}{r} \frac{1}{8} \\ \underline{3\frac{1}{4}} \end{array}$

8. $\frac{5}{6} + \frac{2}{3}$ 9. $\frac{7}{8} + 1\frac{3}{4}$ 10. $3\frac{1}{3} + 5\frac{1}{6}$

11. Two boy scouts went hiking. They hiked $1\frac{2}{5}$ miles before breakfast and $3\frac{3}{4}$ miles after breakfast. How far did they hike altogether?

$$\begin{array}{r} 1.4 \\ \underline{3.75} \\ 5.15 \end{array}$$

Write the fractions as decimals and add as you did before. It is not necessary to read your answer as a common fraction unless it is an equivalent which you know. Distances are often given

in mil[illegible] and hundredths. Here .15 of a mile is
the same part of a whole mile as 15¢ is of a dollar.

Add in the same way as you did in Example 11.

12. $\begin{array}{r}3\frac{1}{2}\\ \underline{1\frac{3}{5}}\end{array}$ 13. $\begin{array}{r}1\frac{4}{5}\\ \underline{6\frac{1}{4}}\end{array}$ 14. $\begin{array}{r}\frac{1}{4}\\ \underline{12\frac{3}{5}}\end{array}$ 15. $\begin{array}{r}\frac{4}{5}\\ \underline{\frac{3}{4}}\end{array}$ 16. $12\frac{1}{2} + 10\frac{1}{5}$

17. James worked $4\frac{3}{4}$ hours last Saturday and $6\frac{5}{6}$ hours the Saturday before selling papers. How many hours did he work on the two Saturdays?

$$\begin{array}{r}4.75\\ \underline{6.833}\\ 11.583\end{array}$$

Write the decimal equivalents and add as before. If this answer were in dollars and cents we would read it as $11.58. This is called rounding the answer to the nearest cent or hundredth of a dollar. As hundredths are more easily understood and are also close enough for our work, we may round our answers to hundredths in reading them when there are three decimal places in our answer.

Add and read the answers to the nearest hundredths:

18. $\begin{array}{r}1\frac{2}{5}\\ \underline{2\frac{1}{3}}\end{array}$ 19. $\begin{array}{r}6\frac{5}{6}\\ \underline{1\frac{4}{5}}\end{array}$ 20. $\begin{array}{r}4\frac{3}{5}\\ \underline{\frac{1}{3}}\end{array}$ 21. $\frac{5}{6} + \frac{3}{5}$ 22. $\begin{array}{r}15\frac{3}{8}\\ \underline{24\frac{1}{6}}\end{array}$

23. $\begin{array}{rl}1\frac{4}{5} = & 1.8\\ 2\frac{1}{6} = & \underline{2.167}\\ & 3.967\end{array}$

Here our answer to the nearest hundredth is 3.97 because .967 is nearer to .97 than to .96.

Note: Let students experimentally determine this by adding .007 to .96 and by subtracting .003 from .97.

24. $\begin{array}{r}1\frac{5}{6}\\ \underline{6\frac{3}{8}}\end{array}$ 25. $\begin{array}{r}6\frac{3}{4}\\ \underline{\frac{1}{6}}\end{array}$ 26. $\begin{array}{r}16\frac{3}{4}\\ \underline{2\frac{2}{3}}\end{array}$ 27. $\begin{array}{r}6\frac{1}{3}\\ \underline{4\frac{5}{8}}\end{array}$ 28. $\begin{array}{r}3\frac{5}{6}\\ \underline{8\frac{1}{4}}\end{array}$

29. Round off to the nearest hundredth:

.166 .334 .833 .666 .167 .333
.834 .667 1.501 4.334 3.999 6.068

30. After selling $3\frac{2}{3}$ apple pies and $4\frac{1}{6}$ cherry pies how many pies had Harry sold altogether?

$$\begin{array}{r}3.667\\ \underline{4.167}\\ 7.834\end{array}$$

Adding we get 7.834. After rounding this to the nearest hundredth we get 7.83. Which of the decimal equivalents when rounded to nearest hundredth = .83?

If we wish then we can call this answer $7\frac{5}{6}$. You will remember that the most difficult group of decimal equivalents was the $\frac{1}{3}$'s and $\frac{1}{6}$'s. This is so because they did not come out as exact decimals but had to be rounded. When adding thirds and sixths, therefore, the third or last figure in the decimal answer is not certain. It is therefore best, if we want the common-fraction equivalent, to round the answer to the nearest hundredth as was done here, and then give the common fraction.

31. Round the following decimal equivalents to the nearest hundredths:

.167, .833, .667, .333, .166, .834, .666, .334.

32. Add: $\begin{array}{r} 6\frac{1}{3} \\ \underline{7\frac{5}{6}} \end{array}$ Round your answer to the nearest hundredth. What common fraction is this the equivalent of? Read your answer with a common fraction.

Add and give common-fraction equivalents to your answers for the following:

33. $\begin{array}{r} 1\frac{2}{3} \\ \underline{\frac{5}{6}} \end{array}$

34. $\begin{array}{r} 1\frac{1}{3} \\ 2\frac{1}{6} \end{array}$

35. $\begin{array}{r} 6\frac{2}{3} \\ \underline{1\frac{1}{6}} \end{array}$

36. $\begin{array}{r} 2\frac{1}{3} \\ 5\frac{5}{6} \\ \underline{6\frac{1}{3}} \end{array}$

37. $\begin{array}{r} 1\frac{2}{3} \\ 5\frac{5}{6} \\ \underline{3\frac{2}{3}} \end{array}$

Having now learned to add common fractions by means of decimals there is nothing new to learn in subtraction of fractions if you know whole-number subtraction. If you have forgotten how to subtract with whole numbers, this will give you a splendid opportunity for reviewing them.

38. Helen sold $3\frac{3}{4}$ yards from a bolt of cloth containing $25\frac{1}{2}$ yards. How many yards remained?

$$\begin{array}{r} 25.5 \\ \underline{3.75} \\ 21.75 \end{array}$$

Note: In order to subtract there is nothing new to learn here. You simply write your fractions as decimals and then subtract as in whole numbers. The answer is $21\frac{3}{4}$ yards.

39. Likewise subtract the following: $\begin{array}{r} 16\frac{5}{8} \\ \underline{4} \end{array}$

Also:

40. $\frac{2}{3} - \frac{1}{6}$

41. $\begin{array}{r} 6\frac{7}{8} \\ \underline{3\frac{3}{4}} \end{array}$

42. $\begin{array}{r} 7\frac{1}{4} \\ \frac{3}{8} \end{array}$

43. $\begin{array}{r} 6\frac{1}{8} \\ \underline{3\frac{5}{6}} \end{array}$

44. $\begin{array}{r} 10 \\ \underline{3\frac{1}{8}} \end{array}$

You are now ready to try the first practice exercise to see how many you can work correctly alone.

Practice Exercise I:

Add:

1. $\begin{array}{r}\frac{3}{4}\\ \underline{10\frac{1}{8}}\end{array}$
2. $\begin{array}{r}7\frac{1}{4}\\ \underline{3\frac{2}{3}}\end{array}$
3. $\begin{array}{r}6\frac{7}{8}\\ \underline{4\frac{1}{3}}\end{array}$
4. $\begin{array}{r}\frac{5}{6}\\ \underline{1\frac{5}{8}}\end{array}$
5. $\begin{array}{r}2\frac{4}{5}\\ \underline{1\frac{1}{2}}\end{array}$
6. $\begin{array}{r}6\frac{2}{3}\\ \underline{\frac{1}{6}}\end{array}$
7. $\begin{array}{r}6\frac{1}{8}\\ 5\frac{5}{6}\\ 3\frac{3}{4}\end{array}$

8. Subtract: $\begin{array}{r}8\frac{1}{3}\\ \underline{3\frac{3}{8}}\end{array}$

9. Subtract: $\begin{array}{r}11\\ \underline{6\frac{5}{6}}\end{array}$

10. Frank ate $\frac{1}{6}$ of a pie for lunch and $\frac{1}{3}$ of it for dinner. What part of a pie did he eat altogether?

Practice Exercise II:

Add:

1. $\begin{array}{r}6\frac{2}{3}\\ \underline{9\frac{5}{6}}\end{array}$
2. $\begin{array}{r}11\frac{1}{2}\\ \underline{\frac{4}{5}}\end{array}$
3. $\begin{array}{r}4\frac{5}{6}\\ \underline{1\frac{1}{2}}\end{array}$
4. $\begin{array}{r}\frac{1}{2}\\ \underline{1\frac{2}{3}}\end{array}$
5. $\begin{array}{r}6\frac{3}{4}\\ \underline{1\frac{5}{8}}\end{array}$
6. $\begin{array}{r}5\frac{5}{6}\\ \underline{2\frac{1}{4}}\end{array}$
7. $\begin{array}{r}8\frac{1}{4}\\ 7\frac{1}{2}\\ \underline{9\frac{7}{8}}\end{array}$

8. Subtract: $\begin{array}{r}7\frac{3}{4}\\ \underline{\frac{7}{8}}\end{array}$

9. Subtract: $\begin{array}{r}12\frac{3}{4}\\ \underline{9\frac{2}{3}}\end{array}$

10. Katherine walks $3\frac{3}{4}$ blocks to school. Her friend Alice walks $5\frac{1}{2}$ blocks. How much farther than Katherine does Alice walk?

At the end of the semester, on June 8, the following test in addition and subtraction of common fractions was given to the 5B group and also to another group of regular 5A students of the same school. The 5A group was chosen as the control group because they had had addition and subtraction of common fractions during this same year. When taking the test the 5B group used decimal equivalents and the 5A group used the customary lowest common denominator method. In fairness to the common fraction group, least common denominators were limited to 24 according to the course of study.

On the same day each group was given an intelligence test (McCall's Multi-Mental Scale) to make comparisons more reliable. All of the tests were given by the writer. Tables VI and VII give the results of the tests in each group.

TEST IN ADDITION AND SUBTRACTION OF COMMON FRACTIONS

Add in the following four examples:

(*Use space at right of each example to work in.*)

1.
3½
4¼

2.
7⅓
1⅞

3.
5⅒
6¾

4.
2¾
⅚

Subtract in the following six examples:

(*Use space at right of each for working.*)

5.
4¾
1⅛

6.
12⅔
9

7.
10
5⅜

8.
9⅔
6¾

9.
6⅔
⅞

10.
⅚
¾

TABLE VI

RESULTS OF COMMON FRACTIONS TEST

Group	N	Mean Accuracy, in Per Cent	$P.E._m$	Mean Time, in Minutes	$P.E._m$	Range in Accuracy	Range in Time
Experimental							
5B	46	94.3 ± 9.2	1.38	6.77 ± 1.6	.24	30 to 100%	3′ 10″ to 15′ 00″
Control							
5A	39	75.3 ± 14.8	2.37	19.63 ± 5.73	.92	10 to 100%	7′ 10″ to 33′ 00″
Difference		19		12.86		20	14′

TABLE VII

RESULTS OF INTELLIGENCE TEST

Group	Mental Ages			Intelligence Quotients		
	Mean	P.E.$_m$	Range	Mean	P.E.$_m$	Range
Experimental						
5B	11.74 ± .91	.13	9 − 1 to 15 − 5	106.0 ± 12.6	1.87	61 to 157
Control						
5A	12.12 ± .91	.14	10 − 0 to 15 − 5	106.1 ± 10.7	1.73	63 to 157
Difference	4½ mo.		11 mo.	.1		2

From the above results the P.E. of the difference of the means was computed from the formula, $\text{P.E.}_d = \sqrt{\text{PE}_{m1}^2 + \text{PE}_{m2}^2}$, and found to be for accuracy, $\sqrt{1.38^2 + 2.37^2}$, or 2.74 per cent, and for time, $\sqrt{.24^2 + .92^2}$, or .95 minutes.

Actual difference between the means for accuracy = 19. per cent. Actual difference between the means for time = 12.86 minutes. For accuracy the ratio of the difference to its P.E.= 19/2.74 = 6.9. For time the ratio of the difference to its P.E. = 12.86/.95 = 13.5. Both of these results are highly significant and mutually reënforcing. When the time taken to prepare the two groups is considered the results become more significant.

It is interesting to note that the student of lowest I.Q. (61) in the decimal group had an accuracy score of 70 per cent, completing correctly all but the three most difficult examples—4, 7, and 10. All of her errors were of the same nature and consisted in not remembering the decimal equivalent.

The student of lowest I.Q. (63) in the common-fraction group had an accuracy score of 40 per cent, completing correctly only Examples 1, 2, 5, and 6. It will be noted that these are the easiest in each list. It should be added also that the former student did hers in 9 minutes, whereas the latter took 28 minutes. This seems to show that the decimal method is not too difficult for dull students to learn but that the lowest-common-denominator method [illegible] be.

This experiment was repeated in another 5B class one [illegible] later in the same school under the same conditions, using the same method and length of teaching as in 1933. As a control group, if it can be so called, the best class in the school, the highest ranking

6A group, was chosen this time. This 6A class was 3 semesters in advance of the experimental group and its average mental age was 1 year 5 months greater than that of the experimental group. The same test was given both groups as before, the 5B group using decimal equivalents and the 6A group the lowest-common-denominator method. Intelligence tests were given both groups as before. The results are shown in Tables VIII and IX.

TABLE VIII

RESULTS OF COMMON-FRACTIONS TEST—SECOND EXPERIMENT

Group	*N*	Mean Accuracy, in Per Cent	P.E._m	Mean Time, in Minutes	P.E._m	Range in Accuracy	Range in Time
Experimental							
5B	44	96.14 ± 10.0	1.5	11.25 ± 2.44	.37	50 to 100%	5 to 21′
Control							
6A	40	88.5 ± 11.6	1.8	14.68 ± 3.33	.52	40 to 100%	5 to 25′
Difference		7.64		3.43		10%	4′

TABLE IX

RESULTS OF INTELLIGENCE TEST—SECOND EXPERIMENT

Group	MENTAL AGES			INTELLIGENCE QUOTIENTS		
	Mean	P.E._m	Range	Mean	P.E._m	Range
Experimental						
5B	11.41 ± .92	.14	9 – 5 to 14 – 8	106.6 ± 12.0	1.81	71 to 149
Control						
6A	12.83 ± 1.13	.18	10 – 6 to 16 – 5	106.5 ± 11.4	1.81	82 to 151
Difference	1 yr. 5 mo.		8 mo.	.1		9

From the results of Table VIII the P.E. of the difference of the means was computed and found to be: For accuracy, $\sqrt{1.5^2 + 1.8^2}$, or 2.34, and for time, $\sqrt{.37^2 + .52^2}$, or .64. Actual difference between means for accuracy = 7.64 per cent. Actual difference between means for time = 3.43 minutes. For accuracy the ratio of the difference to its P.E. $= \frac{7.64}{2.34} = 3.26$. For time the ratio of the difference to its P.E. $= \frac{3.43}{.64} = 5.37$.

These results are again statistically significant. In view of the fact that the control group was one and one-half semesters in advance of the experimental group and their mental age one year five months older, the results of this experiment was a surprise to both the writer and the teacher of the 5B group. It should be added that neither teacher of the 5B nor the 6A group knew in advance that the results of her work were to be compared. It was felt that each would work more naturally if unaware of any competition. The time for the whole experiment was accurately checked for the 5B group, in order to assure that it would be less than that spent in the control group. No one can doubt that the time spent on this topic during the two years in the fifth and sixth grades was greater than that spent in the 5B semester alone. It should be remembered also that 6 weeks of this time in the 5B grade was devoted to learning the meaning of decimals and how to add and subtract them.

Since the mean I.Q. (106.5) of each group was somewhat above average, some may offer the criticism that such an experiment will not give these results except with superior groups. In anticipation

TABLE X

Results in Accuracy and Time, from the Common-Fractions Test Taken by the Five Students of Lowest I.Q. in Each Group

5B Group

Pupil	M.A.	I.Q.	Accuracy	Time
F. P.	9–8	71	100	12′ 10″
C. G.	10–0	77	100	10′ 50″
B. G.	9–8	82	100	15′ 00″
W. T.	9–8	82	100	9′ 10″
A. T.	9–5	83	100	7′ 20″
Mean	9–8.2	79	100	10′ 54″

6A Group

Pupil	M.A.	I.Q.	Accuracy	Time
I. L.	10–2	82	100	21′ 00″
R. B.	11–4	84	80	11′ 50″
H. M.	10–8	85	90	10′ 00″
F. L.	11–6	87	70	16′ 50″
I. S.	11–2	88	70	8′ 40″
Mean	10–9.6	85.2	82	14′ 52″

of this criticism and also to satisfy the writer's curiosity the five students of lowest I.Q. from each group were compared as to accuracy and as to time taken for the test. The results are shown in Table X, page 149.

Again the results were rather surprising and show that the decimal method of adding and subtracting common fractions is not too difficult for dull students.

IV. Conclusion

Practical implications. There are three important implications arising from this study, that can be practically justified from the standpoint of insuring economy and efficiency.

First, after the fourth grade pupil has been taught addition and subtraction of whole numbers, it seems an uncalled-for digression to leave these operations, none of which have been mastered by this time, and to take up an entirely new procedure of learning addition and subtraction of common fractions whose operations are so unlike those of whole numbers. Whereas, if addition and subtraction of decimals are to be taken up in the fifth grade, it is but a simple step from addition and subtraction of United States money with which he has already become familiar in the fourth grade.

Second, the new procedure provides needed review and further learning of addition and subtraction of whole numbers through the addition and subtraction of decimals. That the fundamental processes in whole numbers are not mastered even by fifth, sixth and seventh grade students is shown by an examination of some of the figures from Brueckner's study[11] previously referred to (see pages 132 and 133). In his table, Difficulties in Decimals, he lists the errors that were due to ordinary addition, subtraction, multiplication, and division separate from those due to the decimal situation and separate from all others, for example, omissions, wrong copies, etc. Difficulties in decimals are shown in Table XI.

Table XI shows that 28 per cent of all the mistakes in addition and 74 per cent of all in subtraction of decimals were in addition and subtraction of whole numbers and were not due to the decimal situation. This means that when the errors in addition and subtraction of decimals are taken together half are due to whole-number addition and subtraction. One might well ask, What is the use of trying to learn the operations in both common fractions and

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[11] Brueckner, L. J., *Diagnostic and Remedial Teaching in Arithmetic*, pp. 231-235.

TABLE XI

NUMBER OF ERRORS IN OPERATION WITH DECIMALS
BASED ON A 91-ITEM TEST GIVEN TO 300 STUDENTS IN GRADES 6, 7, AND 8
(ADAPTED FROM BRUECKNER)

	Addition	Subtraction	Multiplication	Division
Due to the basic operation	159	343	465	616
Peculiar to the decimal situation	355	99	1033	2597
All others	66	23	316	538
Total	580	465	1814	3751
Per cent basic of total	28	74	26	16

decimals when whole-number addition and subtraction is only half learned?

A third and most important implication is found in the facility with which remedial work can be done on errors made in addition and subtraction of common fractions when it is done by decimal equivalents.

When a test is scored the customary procedure is to mark the wrong examples on each paper. It could be very valuable if the teacher could at the same time mark the nature of the error made as well as the error itself. This she cannot do for lack of time. For example, it would take hours to trace the kind of error in long division. The nature of the error in addition and subtraction of common fractions is still more difficult to identify when done in the conventional way. If addition and subtraction are done by decimal equivalents, the error is either in the fundamental operation of addition or subtraction or in the decimal equivalent. A glance at the error reveals its nature so that the kind of error can be designated at the same time that the error is marked. In scoring the last test the writer used two different marks for errors, as will be seen on the sample sheet. A vertical oval, ⬯, designated an error in the fundamental operation of addition or subtraction and a horizontal oval, ⬭, an error in the decimal equivalent. It took less than 15 minutes to score the set of 44 papers in this way. The number of each kind of error is listed in Table XII.

When papers are returned to students these marks can be explained to them and they can immediately go to work to correct their own errors from a table of decimal equivalents on the board. What could be a more simple remedial class procedure?

How are most remedial procedures carried on? The fact of the matter is that they are generally not carried on because the teacher

TABLE XII

NATURE AND DISTRIBUTION OF ERRORS IN COMMON-FRACTIONS TEST MADE BY 5B GROUP

	Number of Errors	Per Cent of All Errors
In addition	1	2.3
In subtraction	15	35.0
In decimal equivalent	23	53.4
In miscopies	4	9.3
Total	43	100.0

has not sufficient time. When they are carried on the work is done outside of school hours because it has to be done individually.

The writer also took the time to identify the kinds of errors made by the 6A group on the last test in common fractions. This was a much larger task which took more than two hours. Many errors could not be identified from the pupil's work. (See the sample sheets on pages 154 and 155.) Brueckner says 20 per cent of errors in common fractions cannot be identified from the pupil's work. The nature of the error could not be marked well on the pupils' sheets. See Table XIII for nature of errors.

The first four of these errors were also found among the addition examples. The remaining ten were peculiar to this subtraction test alone. There is no pretense here of a complete analysis of the errors. The thirteen errors in mixed-number borrowing included at least five different varieties. It merely shows that subtraction of common fractions offers many new difficulties not found in addition, whereas subtraction in decimals offers no new difficulty. It also shows that the task of remedial work in addition and subtraction of fractions is a much more complex affair than when it is done by decimal equivalents. The remedial procedure from this test would have to be individual because pupils could not identify their own errors as was true in the case of the decimal method. Few teachers have time for an individual remedial program, let alone teaching 45 to 50 pupils.

It has now been shown that doing addition and subtraction of common fractions by means of decimal equivalents effects a great economy at three different stages in their mastery: first, in the original learning of new skills, second, in the higher accuracy in the test and the less amount of time taken for the tests, and, third, in the simple remedial program following the tests. At the same time a much greater efficiency has been attained.

TABLE XIII

NATURE OF ERRORS IN COMMON-FRACTIONS TEST MADE BY 6A GROUP

In the 4 Addition Examples	Number of Errors
1. In reducing one fraction to the l. c. d.	5
2. Not identified	2
3. In combining mixed number gotten by adding the fractions to the whole number of the answer	1
4. In reducing improper fractions to mixed numbers	2
5. In adding whole numbers	1
6. In adding the whole number of the answer to the whole part of the mixed number in answer	1
7. In reducing to lowest terms	7
8. Wrong operation performed	1
9. In finding l. c. d.	1
10. Failure to reduce improper fraction in mixed number	1
Total	22

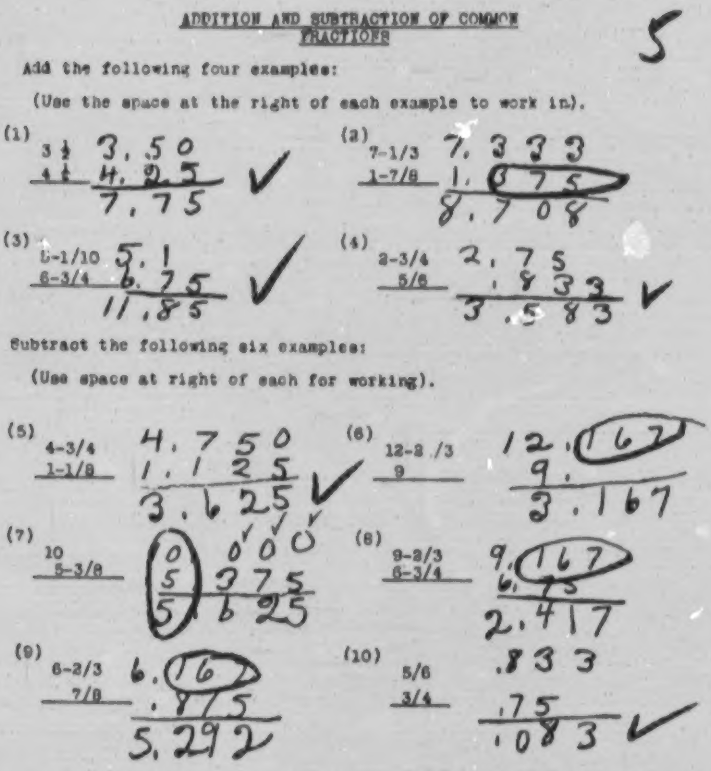
In the 6 Subtraction Examples*	Number of Errors
1. In changing fractions to higher terms of the l. c. d.	9
2. In reducing fractions to lowest terms	3
3. In finding the l. c. d.	4
4. Wrong operation	1
5. In subtracting the whole numbers	1
6. In subtracting numerators	4
7. Calling $10-5\frac{1}{6}=5\frac{1}{6}$	6
8. Forgot the minuend whole number had been borrowed from	1
9. Borrowed in minuend when not necessary	1
10. In mixed-number borrowing	13
11. Subtracted minuend numerator from subtrahend numerator	1
12. Incompleted	1
13. Omitted	1
14. Not identified	8
Total	54

* It should be added that the writer did not count as an error failure to reduce an answer to lowest terms if unreduced answer was correct.

This program allows ample time for multiplication and division of common fractions in Grade 5A and multiplication and division of decimals in Grade 6B. It also allows much time for needed review. There is also sufficient time in the 6A semester for reviews and strengthening of the many needed skills, before entering the junior high school with its important but sadly neglected activity of problem solving.

The problem of economy is not the only issue in this program. When the child is taught his arithmetic in such a way and at such

POOREST TEST PAPER FROM 5B GROUP



ADDITION AND SUBTRACTION OF COMMON FRACTIONS

5

Add the following four examples:

(Use the space at the right of each example to work in).

(1) 3½ + 4¼ — 3.50 / 4.25 / 7.75 ✓

(2) 7-1/3 + 1-7/8 — 7.333 / 1.875 / 8.708

(3) 5-1/10 + 6-3/4 — 5.1 / 6.75 / 11.85 ✓

(4) 2-3/4 + 5/6 — 2.75 / .833 / 3.583 ✓

Subtract the following six examples:

(Use space at right of each for working).

(5) 4-3/4 − 1-1/8 — 4.750 / 1.125 / 3.625 ✓

(6) 12-2/3 − 9 — 12.167 / 9. / 3.167

(7) 10 − 5-3/8 — 0 000 / 5 375 / 5 625

(8) 9-2/3 − 6-3/4 — 9.167 / 6.75 / 2.417

(9) 6-2/3 − 7/8 — 6.167 / .875 / 5.292

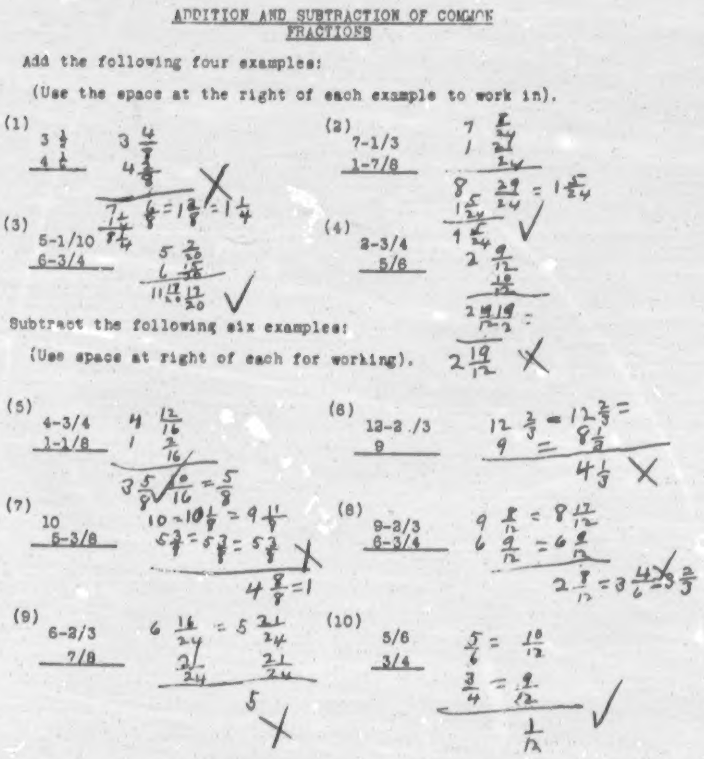
(10) 5/6 − 3/4 — .833 / .75 / .083 ✓

There are 5 correct answers and 5 errors: 1 in subtraction (7) and 4 in decimal equivalent (2), (6), (8), and (9). It will be noticed that the decimal equivalent of ⅔, not being known, caused errors in three examples: (6), (8), and (9).

a time that he understands it and can master it he derives greater satisfaction and joy from it than when the work is only half learned. We must remember that the child learns from his past successes rather than from his past failures, and that self-confidence in his work can only be attained when he succeeds and masters the work as he goes along.

This should be said in conclusion: From reading this article one may be led to believe that the writer is in favor of abolishing the

POOREST TEST PAGE FROM 6A GROUP



ADDITION AND SUBTRACTION OF COMMON FRACTIONS

Add the following four examples:

(Use the space at the right of each example to work in).

(1) $3\frac{1}{2}$ + $4\frac{1}{4}$

$3\frac{4}{8}$
$4\frac{2}{8}$
$7\frac{6}{8} = 1\frac{2}{8} = 1\frac{1}{4}$ ✗

(2) 7-1/3 + 1-7/8

$7\frac{8}{24}$
$1\frac{21}{24}$
$8\frac{29}{24} = 1\frac{5}{24}$
$1\frac{5}{24}$
$9\frac{5}{24}$ ✓

(3) 5-1/10 + 6-3/4

$5\frac{2}{20}$
$6\frac{15}{20}$
$11\frac{17}{20}\ \frac{17}{20}$ ✓

(4) 2-3/4 + 5/6

$2\frac{9}{12}$
$\frac{10}{12}$
$2\frac{19}{12} =$
$2\frac{19}{12}$ ✗

Subtract the following six examples:

(Use space at right of each for working).

(5) 4-3/4 − 1-1/8

$4\frac{12}{16}$
$1\frac{2}{16}$
$3\frac{5}{8}$ ✓ $\frac{10}{16} = \frac{5}{8}$

(6) 12-2 /3 − 9

$12\frac{2}{3} = 12\frac{2}{3} =$
$9 = 8\frac{1}{3}$
$4\frac{1}{3}$ ✗

(7) 10 − 5-3/8

$10 = 10\frac{1}{8} = 9\frac{11}{8}$
$5\frac{3}{8} = 5\frac{3}{8} = 5\frac{3}{8}$
$4\frac{8}{8} = 1$ ✗

(8) 9-2/3 − 6-3/4

$9\frac{8}{12} = 8\frac{17}{12}$
$6\frac{9}{12} = 6\frac{9}{12}$
$2\frac{8}{12} = 3\frac{4}{6} = 3\frac{2}{3}$ ✗

(9) 6-2/3 − 7/8

$6\frac{16}{24} = 5\frac{21}{24}$
$\frac{21}{24}\quad\frac{21}{24}$
5 ✗

(10) 5/6 − 3/4

$\frac{5}{6} = \frac{10}{12}$
$\frac{3}{4} = \frac{9}{12}$
$\frac{1}{12}$ ✓

There are 4 correct answers and 6 errors: 1 in reducing to lowest terms (1); 1 in leaving improper fraction unreduced (4); 1 not identified (6); 3 in borrowing in minuend (7), (8), and (9). The errors in (7), (8), and (9) are not the same although made in connection with borrowing in minuend.

least-common-denominator idea entirely. Such is far from the truth. We need this concept in algebra and in higher mathematics. What the writer is endeavoring to put across to the reader is that since we have evidence that the fifth and sixth grade child does not understand the concept underlying addition and subtraction of common fractions and since the Committee of Seven has found through careful investigation that addition and subtraction of common fractions whose denominators are different cannot be learned efficiently by students of lower mental age than thirteen years, it is better to teach these topics in the seventh, eighth, and ninth grades in a much shorter time and when the student is mentally mature enough to see what they mean and can make them function in his later life. Do not the seventh and eighth grade teachers of the junior high school complain of the students coming to them not knowing how to add and subtract common fractions? Does not the ninth grade algebra teacher do the same? Would it not be better that these teachers teach the least-common-denominator idea for the first time themselves in connection with their own work and at the same time give the students the fundamental principles underlying all fractions? These teachers would then not forever be blaming the teachers below them.

At a recent conference on arithmetic the writer was told by a superintendent of a city system where the placement of arithmetic topics was in accordance with the findings of the Committee of Seven that there was no crowding in the upper grades because of pushing up topics as it took so much less time up there where the maturity of the students gave these topics a ready reception and proper absorption.

CURRENT PRACTICES IN TEACHER-TRAINING COURSES IN ARITHMETIC

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Scope of the investigation. This chapter reports a questionnaire investigation of current practices in teacher-training courses in arithmetic as found in teachers colleges. Copies of the questionnaire were sent to 142 institutions which were members of the American Association of Teachers Colleges (1932–1933). One hundred twenty-nine replies were received. The promptness with which the questionnaire blanks were filled out and returned and the large number of letters which were written supplying additional information indicate that those who are responsible for teacher-training courses in arithmetic are much interested in the investigation. In the paragraphs and tables which follow information is supplied in answer to a series of questions, the chief of which are as follows:

1. What texts are used in training courses in arithmetic?
2. What proportion of the training course is devoted to arithmetic content and what proportion to method?
3. Are prerequisites set up for these courses?
4. Is a course in arithmetic required for teacher certification?
5. To what extent are teacher-training courses in arithmetic offered for the primary and intermediate levels separately and to what extent does one course cover the work for both levels?
6. Are the teacher-training courses in arithmetic classified as mathematics or as education?
7. How large are the classes in training courses in arithmetic?
8. To what extent are demonstration lessons provided? Who teaches the demonstration lessons? When do they occur?
9. What is the importance of a series of devices in the opinions of those who give training courses in arithmetic?

10. What instructional methods are employed in conducting a training course in arithmetic and to what extent is each of these methods used?

11. What are the judgments of those giving courses in the teaching of arithmetic as to the time allotments for the topics of a general three semester-hour course?

Texts used in training courses. Table I lists the principal texts used in teacher-training courses in arithmetic and gives the frequency with which each was reported. All texts reported by two or more institutions are included in this table.

TABLE I

TEXTS USED IN TEACHER-TRAINING COURSES IN ARITHMETIC

Author	Title of Text	Frequency
Brueckner	Diagnostic and Remedial Teaching in Arithmetic	3
Brueckner *et al.*	The Triangle Arithmetic, Book III	3
Brown-Coffman	The Teaching of Arithmetic	5
Klapper	The Teaching of Arithmetic	3
Knight *et al.*	Standard Service Arithmetics	4
Lennes	The Teaching of Arithmetic	2
Lyman	Advanced Arithmetic	2
Morton	Teaching Arithmetic in the Primary Grades	25
Morton	Teaching Arithmetic in the Intermediate Grades	27*
Newcomb	Modern Methods of Teaching Arithmetic	4
Overman	Principles and Methods of Teaching Arithmetic	6
Overman	A Course in Arithmetic for Teachers and Teacher-Training Classes	4†
Roantree-Taylor	An Arithmetic for Teachers	14
Stone-Mallory-Grossmickle	A Higher Arithmetic	9
Strayer-Upton	Arithmetics	2
Taylor	Arithmetic for Teacher-Training Classes	11
Thorndike	The New Methods in Arithmetic	2

* Twenty-three institutions reported the use of both Morton textbooks, 2 use the primary textbook only, and 4 use the intermediate textbook only. Thus, the total number of institutions reporting the use of one or both of these textbooks is 29.

† One institution reported the use of both of the Overman textbooks, 5 use the *Principles and Methods* only, and 3 use the *Course in Arithmetic* only. Thus, the total number of institutions reporting the use of one or both of these textbooks is 9.

As indicated in the notes to Table I, there is some duplication in the frequencies reported in the table due to the fact that some institutions use two of the texts listed. The total number of institutions included in Table I is 102. There remain 27 of the 129 institutions reporting on this topic. Eighteen of these use one each of a miscellaneous series of texts; the remaining 9 use no basal text.

These data were collected in response to a question as to *the* basal textbook for the course. The basal textbook turned out to be two books in some instances, as has been shown. When the question was broadened to include the *number* of basal textbooks used, naturally in many cases a greater number of texts was indicated. One hundred five institutions reported on this item. Of these, 9 indicated that they used no basal textbook, as was stated in the preceding paragraph. Of the remaining 96 institutions, we find 56 using but one basal textbook, 25 using two, 13 using three, and 2 using four.

Content and method. It is possible to get some indication of the relative emphasis placed upon subject-matter content and methods of teaching in teacher-training courses in arithmetic by noting the character of the books listed as the basal textbooks. The reader who is acquainted with the titles listed in Table I can readily draw his own conclusions as to whether the major emphasis is placed upon content or upon method. In the opinion of the writers, Table I may be summarized in this respect by the statement that in approximately half of the institutions reporting the major emphasis of the course is upon content, while in approximately half the major emphasis is upon method. This statement is not based upon a rigorous classification into two categories of the titles given in Table I, and of the 18 miscellaneous titles which were not included in the table, but each of these titles and the frequency with which it occurred were considered in arriving at this conclusion.

It is not possible, however, to determine with sufficient accuracy the relative emphasis placed upon content and methods of teaching in a course simply by noting the title of the basal textbook used. After all, the answer to this question may be obtained only by ascertaining just how the course is organized and conducted. Naturally, the writers have been unable to obtain much detailed information on the manner in which teacher-training courses in arithmetic are organized and conducted. They have obtained from 96 of the institutions from which replies were received a statement of the estimated per cent of the course-time which is given to content and the per cent which is given to method. These per cents are summarized in Table II.

Table II indicates that in one institution the entire time of the

TABLE II

ESTIMATES OF THE PER CENT OF THE TIME OF THE COURSE WHICH IS DEVOTED TO METHOD IN 96 TEACHER-TRAINING INSTITUTIONS

Per Cent Devoted to Method	Frequency	Per Cent
100	1	1.0
75–99	19	19.8
50–74	39	40.6
25–49	19	19.8
1–24	12	12.5
0	6	6.3
Total	96	100.0
Median	57	

teacher-training course in arithmetic is devoted to methods of teaching; that in 19 institutions the proportionate amount of time devoted to method is 75% or more, but less than 100%; etc. It is particularly interesting to observe that in 6 institutions the entire course-time is devoted to content, leaving no time for the development of teaching method. The median amount of time devoted to method is 57% of the total.

It should not be concluded that institutions in which teaching method is stressed in the teacher-training course in arithmetic are necessarily guilty of neglecting the subject-matter preparation of their students. From the data pertaining to prerequisites it is evident that it is a rather common practice for those who are about to enroll or for those who just have enrolled for a course in the teaching of arithmetic to be given a subject-matter arithmetic test to determine whether or not they possess serious deficiencies. Those failing to meet the standard are often required to take a non-credit course in arithmetic content and to pass the subject-matter test before they are permitted to take the teaching course. In some institutions the student failing to pass the subject-matter test takes the non-credit content course and the teaching course concurrently but is not granted credit in the latter until he is able to make a satisfactory score on a test in the former. This plan is followed at Ohio University. The test used is the two arithmetic parts of the Stanford Achievement Test. The standard set is slightly higher than the norm for the tenth grade, the highest norm given in the manual of directions.

Prerequisites. One hundred one institutions answered the question pertaining to prerequisites for the teacher-training course in arithmetic. Thirty of these reported that there were prerequisites for this course; 71 replied that there were none. The nature of the prerequisites in these 30 institutions was not determined.

A certification requirement. One hundred four institutions answered the question whether there was a training requirement in arithmetic for teacher certification. Of this number 84 indicated that an arithmetic course was required for some certificates, while 20 reported that arithmetic was not required for any certificate.

Differentiated courses. In recent years there has been a tendency for teacher-training institutions to develop differentiated courses in arithmetic. Courses designed particularly for the preparation of teachers for the primary grades and for the intermediate grades have appeared on many campuses. Other courses covering the work of all the grades of the elementary school are still found in several institutions.

Table III reveals the practices of 96 institutions as regards general versus differentiated courses. It will be seen that 55 of the 96 institutions offer courses on the primary and intermediate levels separately, while 41 institutions offer the more general type of course. The number of semester hours of credit carried by these courses ranges from 1 to 4 for the differentiated courses and from 2 to 4 for the general courses. There is a slight tendency to grant more credit to the intermediate than to the primary course. Clearly, the modal practice is in favor of 3 hours for a single course, whether differentiated or general.

TABLE III

DIFFERENTIATED VERSUS GENERAL COURSES IN ARITHMETIC, WITH THE NUMBER OF CREDIT HOURS ASSIGNED TO THESE COURSES IN 96 INSTITUTIONS

Type of Course	Hours Credit				Total
	1	2	3	4	
Primary	1	18	31	5	55
Intermediate	1	13	35	6	55
General	0	13	24	4	41
Total	2	44	90	15	151 (96)

Mathematics or education. In a preceding section it was shown that training courses in arithmetic vary greatly in the proportionate emphasis placed upon subject-matter content and methods of teaching. It seems reasonable to assume that a course made up largely of content material would be more likely to be classified as a course in mathematics than would a course in which the major emphasis is upon methods of teaching. At any rate, in answering the question whether arithmetic courses are accepted as mathematics toward a degree, 52 of the 103 institutions reported yes and 51, no.

Replies to the question as to where arithmetic courses are listed in the college catalogue reveal some interesting facts. Table IV summarizes the replies.

TABLE IV

CLASSIFICATION OF ARITHMETIC COURSES IN THE CATALOGUES OF 129 INSTITUTIONS

Classification	Number of Institutions	Per Cent of Institutions
Mathematics	69	53.5
Education	14	10.9
Mathematics and education	8	6.2
Special methods	3	2.3
Socialized mathematics	2	1.6
Not offered on college level	15	11.6
No reply	18	13.9
Total	129	100.0

It will be seen that 69 of the 129 institutions replying to the questionnaire, or 53.5% of the total, classify teacher-training courses in arithmetic as mathematics. Disregarding the 15 institutions which do not offer arithmetic courses on the college level and the 18 institutions from which replies were not received, we have 96 institutions replying to the questionnaire and giving a definite classification to training courses in arithmetic as college courses. Using this new base (96) in calculating per cents, we find that 71.9% classify arithmetic as mathematics, 14.6% as education, 8.3% as mathematics and education, 3.1% as special methods, and 2.1% as socialized mathematics.

Estimates of the per cent of the time of the course which is devoted to method have been reported in Table II. It is pertinent to inquire in this connection whether the per cent of time devoted to method is greater in those institutions which classify the training course in the catalogue as education than in those which classify it as mathematics. The data supplied by the questionnaires were retabulated to answer this question and it was found that in institutions classifying arithmetic as mathematics the mean per cent of the time of the course devoted to method is 50, while in those institutions classifying this subject as education the mean is 63. It will be recalled that the *median* for all institutions is 57 (see Table II).

Class size. Table V is a frequency table showing the size of classes in teacher-training courses in arithmetic in 96 institutions reporting on this item of the questionnaire.

TABLE V

CLASS SIZE IN TEACHER-TRAINING COURSES IN ARITHMETIC IN 96 INSTITUTIONS

Number of Students	Number of Institutions	Per Cent of Institutions
70–79	1	1.0
60–69	0	0.0
50–59	1	1.0
40–49	18	18.8
30–39	45	47.0
20–29	22	22.9
10–19	8	8.3
0–9	1	1.0
Total	96	100.0
Median	33.8	

So far as the writers know, there is no special reason for expecting arithmetic classes to be larger or smaller than classes in other required courses offered to students of the same rank. The large and small classes reported may be due to variations in the student enrollments, to the absence or the presence of prerequisites for the course, or to the existence or the absence of an arithmetic requirement for certification.

Demonstration lessons. Presumably, there are two purposes of a teacher-training course in arithmetic: (1) to correct the student's deficiencies in subject matter and to give him a larger grasp and a better understanding of the subject; and (2) to teach him how to teach the subject to children in the elementary school. The latter purpose is apparently not recognized in the 6 institutions reported in Table II as devoting none of the time of the course to method, and is barely recognized in 12 other institutions which devote less than 25% of the course time to method. Whatever the proportion of time devoted to method in the arithmetic course, it is probable that all graduates of teacher-training institutions who have been trained to teach in the elementary school are required to take a course in student-teaching and that in this course the methodology aspect of arithmetic and other subjects is stressed. It is possible that in those institutions in which but little, if any, of the time of the arithmetic course is given over to method, the student is expected to acquire proper teaching technique in his course in student teaching.

One hundred two institutions reported on whether demonstration lessons were taught in the training school for the classes in arithmetic. Fifty-seven of these 102 institutions reported that such demonstration lessons were provided, while 45 indicated that no such provision was made.

The number of demonstration lessons ranges from 0 to 12, as is indicated in Table VI. It will be seen that the more common practice among those which provide such lessons is to give from 3 to 6 lessons. The median number of lessons is 3.3. The median for those which provide demonstration lessons (excluding the 45 institutions in which no demonstration lessons are provided) is 5.8.

TABLE VI

NUMBER OF DEMONSTRATION LESSONS TAUGHT IN THE TRAINING SCHOOL FOR THE CLASS IN ARITHMETIC, AS REPORTED BY 102 INSTITUTIONS

Number of Lessons Taught	Number of Institutions
11–12	5
9–10	6
7–8	6
5–6	19
3–4	18
1–2	3
0	45
Total	102
Median	3.3
Median of those providing lessons	5.8

The technique of teaching arithmetic is sometimes demonstrated by the teacher of the college class in arithmetic who brings a pupil before his class for this purpose. Of the 90 institutions reporting on this subject, 30 institutions indicated that they used a pupil for demonstration purposes regularly, 15 that they used a pupil occasionally, and 45 that they never used pupils in this manner.

It has been stated that 57 institutions reported that demonstration lessons were taught in the training school for the classes in arithmetic. When asked whether demonstration lessons were taught by the critic teacher, the class teacher, or both, the replies were as follows: critic teacher, 41; class teacher, 5; and both, 11; total, 57.

It is clear that some of those who give teacher-training courses in arithmetic prefer to give their own demonstration lessons, but that these are in the minority.

Another question had to do with the time when the demonstration lesson should be given. Should a demonstration lesson illustrating a certain phase of arithmetic be given before or after the development of the unit in the teacher-training class? On this point, the 57 institutions providing demonstration lessons may be classified as follows: Before the development of the unit, 1; after the development of the unit, 41; before and after the development of the unit, 8; no reply, 7; total, 57.

Clearly, the majority of opinion on this subject is in favor of providing the demonstration lesson after the development of the unit in the teacher-training class.

Table II reported estimates of the proportionate amount of the time of the course devoted to content and to methods of teaching and in a preceding section it has been shown that institutions classifying the course as education give more time to method than do institutions classifying the course as mathematics. When the data were tabulated so as to show the relationship between the proportion of the course devoted to method and the number of demonstration lessons given, it was discovered that, in general, the greater the amount of time devoted to method, the greater the number of demonstration lessons. This was shown by two separate tabulations.

First, the mean number of demonstration lessons was calculated for each of the groups reported in Table II. The results are given in Table VII. It will be seen that there is a steady decline in the mean number of demonstration lessons as the per cent of time devoted to method decreases.

Second, the institutions were classified into three groups according to the number of demonstration lessons given. The first group included those giving no demonstration lessons; the second group,

TABLE VII

Relation between the Proportion of the Time of the Course Devoted to Method and the Number of Demonstration Lessons

Per Cent Devoted to Method	Frequency	Mean Number of Demonstration Lessons
100	1	12.0
75–99	19	3.4
50–74	39	2.6
25–49	19	2.3
1–24	12	0.6
0	6	0.5
Total	96	

those giving from 1 to 5 demonstration lessons; the third group, those giving 6 or more demonstration lessons. The mean per cent of time devoted to method was found to be 41 for the first group, 58 for the second group, and 67 for the third group.

Another tabulation was made to show the mean number of demonstration lessons in institutions classifying the arithmetic course as education and the number in those classifying the course as mathematics. In institutions in which the arithmetic course is called education, the mean number of demonstration lessons was found to be 5.2. In those institutions in which the course is classified as mathematics, the mean number of demonstration lessons is 2.0.

The use of devices. An effort was made to determine the opinion of those who give training courses in arithmetic as to the value of certain devices. The list of devices given in Table VIII was submitted with the request that each be rated for importance. Ratings were to be assigned as follows: A, very important; B, important; C, fair; D, not important. Not all the devices were rated by an equal number of teachers, as will be seen by examining the total column of Table VIII.

In filling out the questionnaire, each teacher was requested to rate each device by using one of the four letters, A, B, C, D, as indicated. After the replies had been tabulated it was decided to assign numerical values to the ratings as follows, A, 3; B, 2; C, 1; D, 0. The purpose of this was to make it possible to calculate a mean rating for each device. The calculated means, given in the

TABLE VIII

DISTRIBUTION OF RATINGS OF DEVICES BY TEACHERS OF TRAINING COURSES IN ARITHMETIC

Device	Rating				Total	Mean Rating
	A=3	B=2	C=1	D=0		
Tests	58	25	3	1	87	2.6
Graphs	25	36	19	4	84	2.0
Flash cards	22	35	19	5	81	1.9
Games	13	31	35	5	84	1.6
Notebooks	12	19	19	24	74	1.3
Crutches	7	23	33	23	86	1.2
Number pictures	4	17	40	12	73	1.2
Rhymes	2	8	34	37	81	0.7
Puzzle problems	0	3	24	55	82	0.4

last column of Table VIII, will indicate the approximate relative importance of these devices in the opinion of those who filled out the questionnaire.

Methods of instruction. What instructional methods are employed in conducting a training course in arithmetic and to what extent is each of these methods used? To obtain an answer to this question, the first seven methods listed in Table IX were named in the questionnaire, with the request that the per cent of the total time devoted to each be indicated. The last two methods named, Nos. 8 and 9, in Table IX, were added by 20 and 7 teachers, respectively. The reader should not conclude that problem solving and tests are considered unworthy of mention by a larger number of teachers. No doubt more teachers would have assigned time to these methods had they been listed in the questionnaire.

All per cents were given in multiples of 5. Table IX gives the distributions of time allowances and the mean time allowance for each method.

The reader will be impressed with the marked variations in practice as regards the use of these instructional methods. The range in amounts of time taken up by the discussion method is from 10% to 50%; by the lecture method it is from 5% to 60%; etc. Whether these figures represent actual practices, however, is a question. They may represent merely the ideals to which these teachers hold. It is difficult to justify such extreme variation, however, either in practice or ideals.

TABLE IX

NUMBER OF INSTITUTIONS DEVOTING VARIOUS PER CENTS OF TIME TO NINE METHODS OF INSTRUCTION IN THE TEACHING OF ARITHMETIC

Method	Per Cent												Total	Mean Per Cent
	5	10	15	20	25	30	35	40	45	50	55	60		
1. Discussion		15	5	13	17	10	4	15		18			97	20.4
2. Lecture	13	32	3	9	10	6	1	2		7		2	85	19.1
3. Demonstration lessons	17	22	9	3	2		1						54	10.8
4. Reports	28	22	7	8	4	1							70	10.8
5. Questions and answers	12	26	10	15	8	3	1	1					76	14.9
6. Projects	27	7	1	2		1	1						39	8.3
7. Case studies	12	14			1								27	8.3
8. Problem solving	4	3	3	3	1	1	1			3		1	20	19.8
9. Tests	3	4											7	7.9

Judgments of a suggested course. In order that definite opinions on time allotments to various topics in a three-hour course in the teaching of arithmetic might be collected, the topics listed in Table X were included in the questionnaire. The 54 hours presumably available in a three-semester-hour course were arbitrarily divided up among the topics as indicated in the column headed "Hours." Those replying to the questionnaire were asked to indi-

TABLE X

JUDGMENTS OF TEACHERS OF TRAINING COURSES IN ARITHMETIC OF SUGGESTED TIME ALLOTMENTS FOR A THREE-SEMESTER-HOUR COURSE

Suggested Topics	Hours	Teachers Voting for						Total
		Less Time		No Change		More Time		
		No.	%	No.	%	No.	%	
Equations	3	24	43	24	43	8	14	56
Devices	3	30	50	23	38	7	12	60
History of arithmetic	6	55	74	16	22	3	4	74
Objectives of arithmetic	6	40	56	24	33	8	11	72
Problem solving	12	26	37	20	29	24	34	70
Fundamental operations	18	32	45	26	36	14	19	72
Drill exercises	1	3	5	25	44	29	51	57
Common fractions	1	3	5	22	37	34	58	59
Decimal fractions	1	4	7	23	38	33	55	60
Percentage	1	4	7	17	30	36	63	57
Interest	1	7	12	22	38	29	50	58
Mensuration	1	5	9	24	42	28	49	57

cate whether they considered the proposed time allotments insufficient, adequate, or more than adequate.

It will be seen that those who would give less time outnumbered those who would give more time to equations, devices, history, objectives, problem solving, and the fundamental operations, although the vote is very close in the case of problem solving. On the other hand, those who would give more time than was suggested in the questionnaire decidedly outnumbered those who would give less time in the case of drill exercises, common fractions, decimal fractions, percentage, interest, and mensuration. Evidently, many consider one hour inadequate for these topics.

Some teachers suggested additional topics for inclusion in a course in the teaching of arithmetic. These included: banking, partial payments, the metric system, ratio and proportion, painting and paper-hanging, graphs, checks and drafts, workbooks, budgets, investments and savings, insurance, test construction, taxes, and others.

The writers are presenting the data of Table X for such interest and informational value as they may possess. They are not recommending a course made up of the topics listed, nor are they endorsing the suggested time allotments. It will be noted also that the course upon which these teachers voted is a general course and that no judgments were collected relative to allotments of time to the topics included in courses differentiated for the primary and intermediate grades.

Summary. A questionnaire investigation of current practices in teacher-training courses in arithmetic among members of the American Association of Teachers Colleges brought to light the information summarized in the following paragraphs:

1. Thirty-five different publications are used as textbooks, counting a series of elementary school arithmetic textbooks as one publication. Twenty of these are primarily content books; 15 are books devoted largely to the teaching of arithmetic. Of the 120 institutions reporting the use of a textbook, 58 use a book which is largely content, while 62 use a book which is largely method. The textbooks which are used in two or more institutions are listed in Table I.

2. As reported by 96 institutions, the per cent of time devoted to method in teacher-training courses in arithmetic ranges from 0 to 100. There are 6 institutions at the former extreme and 1 at the

latter. The median per cent of time devoted to method is 57. Some institutions, in which a relatively large amount of the time of the course is devoted to method, require students to make good the more serious subject-matter deficiencies in specially organized classes in which the work is done without college credit.

3. Of 101 institutions answering a question pertaining to prerequisites, 30 indicated that they have prerequisites for this course and 71 that they have none.

4. Of 104 institutions answering a question as to whether there was a training requirement in arithmetic for teacher certification, 84 indicated that there was such a requirement and 20 that there was not.

5. Of 96 institutions reporting on the subject of general and differentiated courses in arithmetic, 55 indicate that they offer courses on the primary and intermediate levels separately, while 41 offer only the more general type of course.

6. The number of semester-hours' credit carried by training courses in arithmetic ranges from 1 to 4. Sixty per cent of the 151 courses reported by 96 institutions carry three semester hours of credit.

7. In 52 of 103 institutions reporting, arithmetic courses are accepted as mathematics toward a degree, while in 51 institutions arithmetic is not classified as mathematics for this purpose.

Of 96 institutions reporting upon the classification of arithmetic courses in catalogues, 69 indicate that it is classified as mathematics, 14 as education, 8 as mathematics and education, 3 as special methods, and 2 as socialized mathematics.

More of the course time is devoted to method in institutions classifying arithmetic as education than in institutions in which it is classified as mathematics.

8. Classes in courses in the teaching of arithmetic range in size from less than 10 to more than 70. The median for these classes is 33.8.

9. Of 102 institutions reporting as to whether demonstration lessons were taught in the training school for the classes in arithmetic, 57 indicated that such lessons were provided while 45 indicated that they were not. The number of demonstration lessons ranges from 1 to 12 with a median of 5.8.

Of 90 institutions reporting, 30 indicated that they used a pupil before the class for demonstration purposes regularly, 15 that they used a pupil occasionally, and 45 that they never used a pupil for this purpose.

In the 57 institutions in which demonstration lessons were taught in the training school, these lessons were taught by the critic teacher in 41 institutions, by the class teacher in 5, and by both the class teacher and the critic teacher in 11.

Of 50 institutions reporting, 41 have the demonstration lesson *after* the development of the unit in class, one *before* the development of the unit, and eight before and after the development of the unit.

There is a positive and clearly defined relationship between the per cent of the course time devoted to method and the number of demonstration lessons. Institutions classifying the course as education provide more demonstration lessons than do institutions calling the course mathematics.

10. Among the devices rated, tests, graphs, and flash cards were considered to be of relatively large worth; games, notebooks, crutches, and number pictures were given an intermediate rating; and rhymes and puzzle problems were thought to be of relatively little value.

11. Marked variability is shown in the proportion of class time which is taken up by various methods of instruction. The most popular instructional methods with the means of the per cents of class time devoted to each are: discussion, 29.4%; problem solving, 19.8%; lecturing, 19.1%; questions and answers, 14.9%; demonstration lessons, 10.8%; and reports, 10.8%.

12. Judgments of suggested time allotments for a three-semester-hour course showed much difference of opinion. There was a definite disposition to devote less than six of the fifty-four class hours to the history of arithmetic and less than 6 hours to the objectives of arithmetic. Other tendencies may be gleaned from data found in Table X.

Concluding statement. In conclusion, the writers wish to point out that the study which they have made supplies considerable information relative to prevailing practices in the training of teachers of arithmetic but does not indicate which of these practices are desirable and which are undesirable. What constitutes the ideal program for training teachers of this subject is not known. Obviously, those who are charged with the responsibility do not agree on how such a program should be planned and executed.

There is a great need for research in order that the merits of various programs may be tested. This research will require much time and all of the skill which the profession can muster. In the meantime, it is worth while for those who conduct the training courses to know what others are doing. One should at least justify to himself any of his practices which are out of harmony with current tendencies in teacher training.

THE PROBLEM OF TRANSFER IN ARITHMETIC

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The Problem

When a new educational theory, or procedure, is first introduced it usually meets with considerable opposition. Although a few teachers and administrators take up every new fad just because it is new, the great majority are conservative and slow to change their ideas and habits. This period of inertia, or opposition to change, is usually followed by one in which the new idea rapidly gains widespread, but uncritical, acceptance. Once convinced that it has merit, educators become enthusiastic and can see neither the weak points of the new nor the strong points of the old. As a result educational theory and practice tend to swing from one extreme to the opposite. Finally comes the period of sober judgment, of critical evaluation of the old and of the new, which usually results in a compromise, a middle-of-the-road course, or a combination of the good points of both the old and the new.

The problem of transfer of training affords an excellent illustration of these three stages. As early as 1890, William James showed that practice in learning one poem produced little or no gain in ability to learn another poem. This pioneer experiment was soon followed by others, by Jastrow, Raif, Thorndike, Woodworth, and numerous others, in all of which the training of one mental function produced little or no gain in related functions. In spite of this experimental evidence, however, the curriculum of the majority of our schools and the methods used by most of our teachers continued for many years to be based upon the old faculty psychology and a belief in general and magical transfer.

Largely through the efforts of Thorndike this naïve faith in transfer was finally shaken, and educators and educational practice rapidly swung to the opposite extreme. "Formal discipline" and "transfer of training" came to be regarded as exploded and discredited theories. As a result, our present curricula and methods of teaching are largely based on the theory that "we teach what we teach, that and no more."

At the present time we are entering upon the third stage, that of critical evaluation. Later experiments, together with a more careful interpretation of the results obtained by the early workers in this field, have shown that a belief in the total absence of all transfer is just as false as the earlier belief in complete and magical spread. This chapter has been written in order to acquaint the rank and file of teachers of arithmetic with the latest evidence on the question of transfer, in the hope that it will help in shortening the period of readjustment and in bridging the gap between our present knowledge of the subject and the prevailing practice in our schools.

EXPERIMENTAL EVIDENCE

Owing to limitations of space only a few of the experiments on the spread of learning can be summarized in this article. Judd [1],* in 1908, reported an experiment by Schalcow and himself. Two groups of fifth and sixth grade boys practiced throwing a dart at an object under water. Before starting practice one group was given a full theoretical explanation of refraction. In the first series of trials, with the object under twelve inches of water, one group learned as rapidly as the other. The depth of the water was then reduced to four inches. The boys without the theoretical training were badly confused and apparently received no help from the former practice. The group with the theory, however, very quickly adapted themselves to the new conditions.

In 1911 Starch [2] reported an experiment with fifteen subjects. These were first given a series of tests covering addition of fractions, adding and subtracting two numbers of three digits each, multiplying two- and four-digit numbers by one-digit numbers, dividing three-digit numbers by one-digit numbers, and memory span for numbers and words. After the preliminary test, eight of the subjects were given fourteen days of practice on mental multiplication of three-digit numbers by one-digit multipliers. All fifteen subjects were then given a different test covering the same points as the preliminary test. All calculations on the tests, as well as in the practice, were performed mentally. There was little change in the memory span for either group. The practiced subjects showed from 20% to 40% more improvement on the arithmetical tests than the unpracticed subjects. Starch concludes that "Training in one type of arithmetical operation improves very considerably the ability to do other fundamental arithmetical operations."

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* The numbers in brackets refer to the bibliography given at the end of this article.

In 1924 Knight and Setzafandt [3] reported a study showing that practice in adding fractions having certain denominators resulted in improvement in adding fractions with other denominators. Two groups, neither of which knew how to add fractions at the start of the experiment, were given instruction and equal amounts of practice on this process. The practice material for the first group contained fractions having 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 21, 24, 28, and 30 as denominators; that for the second group contained only the denominators 2, 4, 8, 10, 12, 16, and 24. Following the period of instruction and practice, both groups were given tests involving all of the denominators. It was found that the unpracticed group added fractions having denominators 3, 5, 7, 9, 14, 15, 18, 21, 28, and 30 almost as well as the group that had practiced with these denominators.

Beito and Brueckner [4], in 1930, conducted an experiment to determine to what extent the teaching of addition combinations in one order carries over to the same combinations in the reverse order. Ninety-three pupils in Grade 2B were taught and practiced on thirty-six reversible addition combinations. In all of the teaching and practice these combinations were met in one order only, the larger number always coming first. The combinations were divided into three groups of twelve each and one week was devoted to the study and practice of each group. Tests were given at the beginning and at the end of each week, covering the reverse form of the combinations, as well as the direct form as taught. The total gain for the direct combinations, which were taught and practiced, was 82% of the possible gain. The total gain for the reverse combinations, which were not studied, was 84.7% of the possible gain. The authors conclude: "When pupils of any mental level are taught only the direct form of an addition combination such as $\begin{array}{r}7\\\underline{4}\end{array}$, as nearly as can be, the reverse form, $\begin{array}{r}4\\\underline{7}\end{array}$, is learned concomitantly at least as completely as the direct form. The bond formed in learning the direct form of an addition combination carries over almost completely to the reverse form."

A somewhat similar experiment was reported by Olander [5] in 1931. Approximately thirteen hundred children in the first half of the second grade took part in the experiment. Part of the pupils were taught all of the one hundred addition and one hundred subtraction combinations; others were taught only fifty-five combinations in each process. At the end of seventeen weeks of instruction and practice all of the pupils were tested on all two hundred combinations. It was found that the pupils who were taught only one hundred ten combinations made almost as good a score on the ninety combinations that they had not studied as the score made by the pupils who had studied these combinations. Olander concludes:

> The ability gained by children on fifty-five simple number combinations in addition and in fifty-five similar combinations in subtraction transferred almost completely to the forty-five remaining simple number combinations in each of the two processes. Between addition and subtraction little significant difference in transfer was found. In subtraction the amount of transfer was only slightly less than in addition.

Olander also attempted to study the effect of generalization on transfer. Some of the classes were given three minutes of instruction each day in generalizing groups of combinations. They were led to see the general law in zero combinations, the relation between a combination and its reverse, and the connection between the addition and subtraction combinations. The remaining classes spent the corresponding three minutes in drill without any attempt at generalization. No significant differences in the amount of transfer were found between the groups taught by these two methods. The author states that this absence of any apparent effect of generalization on the amount of transfer may be accounted for by one or more of the following explanations:

> (*a*) The function practiced was too narrow to necessitate special stress on generalization, that is, the children generalized without help from the teacher.
>
> (*b*) The length of time spent on generalization . . . was too brief.
>
> (*c*) The children were too immature to profit from abstract verbal generalizations.

Coxe [6] attempted to measure the effect of the study of beginning Latin on the ability to spell English words of Latin derivation. In some of the Latin classes participating in the experiment no attempt was made to connect the Latin and the English spellings, in other classes the teachers pointed out the similarity in the spelling of English and Latin words, and in still other classes the teachers not only pointed out the likenesses in spelling but also attempted to develop general rules and principles. The results indicated that Latin, as ordinarily taught, results in little or no improvement in spelling words derived from the Latin, but that the spelling of such words is considerably improved by pointing out similarities between the Latin and the English spellings, and is still further improved by the formulation of general rules and principles which govern spelling.

Overman [7], in 1927–1928, made a study to measure the amount of improvement in the subtraction of two-digit numbers, and in the addition and subtraction of two- and three-digit numbers, resulting from instruction and practice in the addition of two-digit numbers. Pupils in the first half of the second year were given fifteen days of instruction and practice on three types of addition examples: (*a*) the addition of two numbers of two digits each, (*b*) the addition of three numbers of two digits each, and (*c*) the addition of a two-digit, a two-digit, and a one-digit number, in the order named. At the beginning, during, and at the end of the fifteen days the pupils were tested not only on these three types of examples, but on a number of other related types in addition and subtraction. Table I gives the percentage of possible gain on each type.

Table I indicates that the fifteen days of instruction and practice resulted in a gain of 93.8% of the possible gain in adding two numbers of two digits each; 89.0% of the possible gain in adding three numbers of two digits each; and 91.6% of the possible gain in adding a two-digit, a two-digit, and a one-digit number, in that order. Substantial gains were also made on each of the other types of examples, although they were not included in the instruction and practice. In many cases the gain on the untaught types was almost as great as on the types taught. For example, the training on the addition of two numbers of two digits each carried over to the addition of two numbers of three digits and resulted in a gain of 89.8% of the possible gain on that type of example. It

TABLE I

TYPES OF EXAMPLES TAUGHT, WITH PER CENT OF GAIN

	Type of Example	Per Cent of Possible Gain
Taught	2 + 2*	93.8
	2 + 2 + 2	89.0
	2 + 2 + 1	91.6
Not Taught	3 + 3	89.8
	3 + 3 + 3	83.4
	2 + 2 + 2 + 2	81.0
	3 + 3 + 3 + 3	76.1
	2 + 1 + 2	90.4
	1 + 2 + 2	73.1
	1 + 2 + 1	63.2
	3 + 2	54.7
	2 + 3	44.8
	3 + 1	61.7
	1 + 3	54.2
	2 + 3 + 1	44.6
	2 − 2	77.4
	3 − 3	75.7
	2 − 1	57.2
	3 − 2	46.2
	3 − 1	54.1

* This notation represents the addition of two numbers of two digits each.

also produced 77.4% of possible gain on the subtraction of two-digit numbers, and 75.7% of possible gain on the subtraction of three-digit numbers.

In order to determine the effect of the method of teaching on the amount of transfer, the pupils were divided into four matched groups of 112 pupils each. These groups were taught by four different methods as follows:

Method A. The pupils were shown how to perform the operation without any attempt to form generalizations or to teach underlying principles.

Method B. The pupils were helped to formulate general methods of procedure from the particular types taught. This will be referred to in what follows as the *method of generalization.*

Method C. The reasons and principles underlying the methods of procedure taught were considered with the pupils. The formulation of general rules of procedure was avoided as much as possible. This will be referred to as the *method of rationalization.*

Method D. The pupils were encouraged to formulate general methods of procedure and the underlying principles were also discussed.

The four methods of teaching were found to produce practically equal amounts of transfer to those types of examples that did not involve the difficulty of placing numbers having different numbers of digits. The relation of these types to the types taught was apparently so close that the transfer took place without conscious generalization, or the pupils were able to make this generalization without help. In the case of examples involving the placement of numbers having different numbers of digits, the generalization alone increased the amount of transfer by 45.1%, the rationalization alone increased it by 15.5%, and the two combined increased it by 36.9%. The connection between the examples of this type and those taught was apparently less obvious to the pupils, and some of them were unable to make the generalization without assistance. Many of these pupils, however, were able to make the generalization with the assistance of the teacher and the other members of the class. These results would seem to indicate that Olander's first explanation of the fact that the generalization did not increase transfer in his experiment is probably the correct one—the function practiced was so narrow that the pupils generalized without help.

CONCLUSIONS

The experiments summarized above, together with many others not mentioned because of a lack of space, warrant the following conclusions:

1. Improvement in one mental function, through instruction and practice, often results in very substantial gains in other related functions. In some cases this improvement may be as great as that in the function practiced.

2. The amount of transfer, or improvement in the untaught function, depends not only upon the relation between the taught and the untaught functions, but upon the method of teaching as well. The fact that no transfer is obtained by one method of teaching is no proof that considerable spread might not take place with another method of instruction.

3. Transfer is greatly increased, at least in some cases, by methods of teaching that (*a*) help the pupils formulate general rules or methods of procedure from the specific cases taught, (*b*) emphasize

likenesses between the old and the new situation and train the pupils to look for and recognize such likenesses, and (*c*) give the pupils a real understanding of the method of procedure employed, by making clear the reasons, or the principles, underlying this method.

Let us next consider what effect our present knowledge concerning the spread of learning and the methods of increasing such spread should have on the methodology of arithmetic.

1. Having proved that transfer does occur in useful amounts, we must not go to the opposite extreme and expect it to accomplish the impossible. Experimental evidence shows that many pupils, after having learned how to add $\begin{array}{r}32\\ \underline{22}\end{array}$, are able without further instruction to add $\begin{array}{r}232\\ \underline{322}\end{array}$ and to subtract $\begin{array}{r}36\\ \underline{24}\end{array}$ and $\begin{array}{r}435\\ \underline{213}\end{array}$. It also shows, however, that many pupils fail to make this transfer. We cannot, therefore, give instruction and practice on two-place addition alone, and trust to transfer to take care of three-place addition and two- and three-place subtraction. We must continue to teach all of the different facts and all of the different types of examples in the fundamental processes.

2. Although we cannot depend upon transfer alone we must not lose sight of the fact that it can be of great help to us in our teaching. Both economy and efficiency demand that we teach in such a way as to take the greatest possible advantage of this assistance, that we make the securing of the maximum helpful transfer a conscious aim of our instruction. The experimental evidence indicates that this is best done in two ways: (*a*) by helping pupils formulate general principles and general methods of procedure from the specific types taught and (*b*) by pointing out likenesses of principle and method between the new and the old. Thus, in teaching the addition of $\begin{array}{r}232\\ \underline{24}\end{array}$, we should not only show the pupil how to write the numbers but should also see that he forms the generalization that the right-hand column must be kept straight. Then when he comes to addition examples such as $\begin{array}{r}356\\ \underline{4}\end{array}$ and $\begin{array}{r}21\\ 132\\ \underline{5}\end{array}$, and subtraction examples such as $\begin{array}{r}38\\ \underline{5}\end{array}$ and $\begin{array}{r}185\\ \underline{32}\end{array}$ and $\begin{array}{r}156\\ \underline{4}\end{array}$, we must make sure that he sees that the same generalization applies in these new situations. Generalization and

looking for likenesses between the new and the old should become habitual with both teachers and pupils.

UNSOLVED PROBLEMS

There are still a number of important questions concerning the teaching of arithmetic that cannot be answered until we have further experimental evidence concerning transfer.

1. What are the different facts and the different types of examples that must be taught in the fundamental processes? Under the influence of the belief that all learning is specific, the tendency in recent years has been to divide the subject matter of arithmetic into a larger and larger number of smaller and smaller units. Whereas it was once thought sufficient to teach 45 combinations in each of the four processes, or a total of only 180 separate facts, we now teach many more than this number. According to Osburn [8], who gives an exhaustive analysis of the facts involved in the fundamental processes, there are altogether 1,680 facts that the pupils may need to know.

Along with this increase in the number of facts to be learned, from 180 to 1,680, there has been a corresponding increase in the number of steps employed in teaching the processes. Whereas once it was deemed sufficient to teach the pupils how to carry in addition, we now teach them how to carry 1 ten, then how to carry 2 tens, 3 tens, etc. How to carry 1 hundred, 2 hundreds, 3 hundreds, etc. How to carry 1 ten and 1 hundred, 2 tens and 1 hundred, etc., etc. Obviously, if each of these represents a different situation to the pupils, there are an infinite number of combinations and permutations to be learned in carrying alone.

It is quite evident that this division of the subject matter into more and smaller teaching units cannot go on indefinitely. In fact, the experimental evidence lends considerable support to the view that it may have gone too far already, and that some of our analyses of the learning difficulties in the fundamentals of arithmetic may be based on superficial differences which the average pupil does not notice.

Considerably more experimental evidence will be necessary before we can definitely say just what facts, and what steps in the processes are sufficiently different to make it necessary that they be taught as separate units, and not until this evidence is obtained will it be possible to construct a scientific curriculum in the fundamentals of arithmetic. The evidence to date, however, seems sufficient to warrant the prediction that the present tendency toward dissecting arithmetic into smaller and smaller teaching units, based on external and possibly superficial differences, is likely to give way to larger units based on fundamental likenesses of method and of principle. The experimental determination of the unit skills in arithmetic should receive much attention in the near future.

2. Does an understanding of the reasons for a method of procedure help the pupil to generalize that method and to increase his ability to apply it in new and slightly different situations? Does rationalization increase transfer? Judd, in his experiment in training boys to strike an object under water, found that a knowledge of theory was of great assistance in promoting transfer. Overman, however, found that a knowledge of why the right-hand column must be kept straight in adding and in subtracting was of little help to second grade pupils in forming this generalization and in applying it to new situations. This may have been due to the age of the pupils or to the nature of the subject matter. Possibly with older pupils, or with different subject matter, rationalization might greatly increase the spread. It seems reasonable to suppose, for example, that a pupil who knows the short methods of multiplying by 25 and by 12½, and also understands why these methods work, would be more apt to invent the corresponding method of multiplying by 33⅓, than the pupil who knows the first two methods but does not understand the principle involved. Considerably more experimental evidence bearing on the effect of rationalization on transfer, with pupils of different ages and abilities and with different subject matter, must be obtained before any one can attempt to give a final answer to the question of the value of rationalization in teaching arithmetic.

3. Another question which cannot be finally answered in the light of our present experimental evidence is the question of interference, or negative transfer. Most of the studies which have been made of pupils' errors show that many mistakes are due to an attempt to apply to a new situation a method of procedure which was learned in another situation and which is not applicable to the new. Overman, in the experiment previously described, found that many pupils in attempting to add 24 and 322, without instruction, wrote $\begin{array}{l}322\\ \underline{24}\end{array}$ instead of $\begin{array}{r}322\\ \underline{24}\end{array}$. In fact the first plan was more popular than

the second. This was probably due to transfer of the habit of
3 keeping an even margin on the left in all written work. An-
2 other popular method of adding 322 and 24 was to add all of
2 the digits, obtaining 13 as a result. This was probably due
2 to transfer from examples such as that given at the left, with
4 which the pupils were already familiar.

Knight [9] gives a number of illustrations of possible interference due to transfer of wrong procedures.

Many children will perform correctly *before* they have studied multiplication, the example $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$. After they have studied multiplication they will often, if not cautioned, find the answer $\frac{3}{10}$. Here they remember to add, but they now add *both* the numerators and denominators. A possible explanation is that having in recent work in multiplication treated both numerator and denominator alike, as it were, they do similarly in subsequent addition. In general, it may be said that what a child does at any time is a result of all his past experience. Some of his past experience may lead him in error, and many children possess quite limited amounts of ability to analyze from their past experience only the correct and useful things to use in any given situation.

All observant teachers have noticed many similar cases of negative transfer in their own experience. Recently a college freshman, in a class in arithmetic for teachers, obtained $\frac{4}{9}$ as the answer to a problem and changed it to $\frac{2}{3}$. Upon questioning, she said that $\frac{4}{9}$ is equal to $\frac{2}{3}$ because you can do the same thing to both the numerator and denominator of a fraction without changing its value. She had, therefore, extracted the square root of both the numerator and the denominator.

Since many errors are due to negative transfer we must not only teach so as to secure the maximum useful transfer, but we must also teach so as to reduce negative transfer to a minimum. We have little experimental evidence bearing on how the latter may be accomplished. Until such evidence is obtained, the following suggestions may be found helpful.

a) Many cases of negative transfer seem to be due to the failure or the inability of the pupils to analyze the new situation and to recognize fundamental differences between the new and the old, as well as superficial likenesses. In teaching pupils how to add 12.1, .34, and .5 the teacher not only should show the pupils how to add such decimals, but should contrast the addition of decimals with the addition of whole numbers, emphasizing that, whereas in the one case it is the right-hand column that must be kept straight, in the other case it is the decimal points. Emphasizing differences to prevent negative transfer is probably just as important an element in good teaching as pointing out likenesses in order to assist helpful spread.

b) Children constantly generalize, whether we encourage them to do so or not. Many cases of interference are due to errors in generalization. These errors are of two kinds. Pupils often generalize too widely. This is illustrated by the student who, having learned that the numerator and the denominator of a fraction may be multiplied or divided by the same number without changing the value, concluded that she could perform any operation on the numerator and the denominator as long as she treated both terms alike. Pupils often form incorrect generalizations. In the first cases of carrying that the pupils meet, the number to be carried is usually 1. It is not at all uncommon for pupils to generalize from this and to fall into the error of always carrying 1.

It would seem that the best way to avoid errors due to incorrect and too wide generalization is to make training in generalization an important part of our teaching. Plenty of practice should be given and the dangers of generalizing too far and from too few data should be noted. As long as the pupils are permitted to form their own generalizations without supervision or training, we can expect many errors due to negative transfer.

c) Unfortunately we have little or no experimental evidence on the effect of rationalization on negative transfer. One would naturally suppose, however, that the pupil who understands why a certain mechanical method of procedure produces the correct result would be less apt to misapply this method than the pupil who lacked such understanding. Merely warning that in adding we must write 12.1, .34, and .5 one way; and 132, 42, and 3 another should undoubtedly be some protection against applying the same method to both examples. It would seem, however, that an understanding of why the decimal points must be kept in a straight line in the first case, and the right-hand column must be straight in the second, would be an added protection. It is certain that among adults, those who try to apply mechanical methods without a full understanding of the underlying principles fall into many errors. The misuse of statistical methods by many educational workers is sufficient evidence of this fact. It is at least possible that the present tendency towards teaching the fundamentals of arithmetic on a mechanical level, without rationalization, may be a prolific breeder of error; and that the proper kind of rationalization, at the proper time, would be the best protection against mistakes of this character. Further experimental evidence must be obtained on this point before methodology in arithmetic can be placed on a scientific foundation.

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TYPES OF DRILL IN ARITHMETIC

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Defining of word "drill." In its original setting the word "drill" was associated directly with formal military discipline and training. Drill meant the exercise of men in formations and movements and in the execution of commands. From this early beginning the term has been transferred to regular and thorough discipline in many branches of knowledge. In a strictly military program drill was shown to be effective. Overwhelming proof of its need and desirability in such affairs is furnished in every chapter of history from the time of Leonidas to Foch.

It has been apparent for some time that sheer exercise in such activities as the learning of arithmetic, history, language usage, and spelling gives no great assurance of mastery in these fields. Although the early colonial schoolboy, from all accounts and inferences, struggled with his sums in arithmetic for years and confined his efforts to relatively few studies he was still in many respects a novice in these subjects when he entered college. Karpinski relates that arithmetic was taught in the American colleges until well into the nineteenth century.[1] As time passed, the teaching of the subject improved somewhat both in materials and in methods of instruction. However, authoritative comment on the teaching of arithmetic is still strongly colored with doubt about much of the drill used. Thorndike[2] very clearly indicates his dissatisfaction when he states:

> The older methods trusted largely to mere frequency of connection—that is, to mere repetition—in order to form habits of arithmetical knowledge and skill. Pupils said their tables over and over. They heard and said $7+9=16$, $6\times 8=48$, and the like, over and over

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[1] Karpinski, L. C., *The History of Arithmetic*, p. 177. Rand McNally & Co., 1925.

[2] Thorndike, E. L., *The New Methods in Arithmetic*, p. 57. Rand McNally & Co., 1924.

> again, hour after hour, day after day. Yet scores of repetitions did not form the bonds perfectly. A girl who learned to connect the names of 45 children in her class with their faces infallibly in a few weeks from casual incidental training did not learn to connect the 45 addition combinations, 1 + 1 to 9 + 9, with their answers in systematic drills of twice that time. A boy who in two months' vacation learned, from a few experiences of each, to know a thousand houses, turns of paths, flowers, fishes, boys, uses of tools, personal peculiarities, slang expressions, swear words, and the like, without effort, seemed utterly incapable of learning his multiplication and division tables in a school year.

The older methods which Thorndike refers to and criticizes have been by no means discarded *in toto*. One might quite accurately state that some of these methods obtain in a majority of the textbooks now in use. The drills produce little apparent effect and, further, we might well conclude that the reason why pupils learn some things easily, and with lasting effect, while they fail to learn other things no more complex, or forget when they do learn, can largely be accounted for in terms of misuse of the *law of effect*. That is to say, a pupil learns and remembers when it pays him to do so, for example, with his playmates and related objects or activities. However, in matters of less apparent value and immediate use and interest, the marked tendency is mastery of a questionable nature even though dosage is great both in amounts and in frequency of administration.

Kirby, Thorndike, and Hann even went so far as to, by implication, question altogether the effectiveness of drill.[3] Though the results of the experiments of these three men were slightly at variance, they agreed on one point, that drill in arithmetic does produce increased efficiency in the processes receiving exercise. Such experiments have been sufficiently verified, so that there is little question about the desirability of drill in any teaching or learning program. However, some persons are not convinced. Wheeler and Perkins[4] maintain that gains in arithmetical skill are not due to exercise but to maturation of the organism. Doubtless there is some truth in such assertions. Gains probably can be accounted for by both maturation and exercise. However, sheer physical maturation without exercise of proper neural patterns could not bring into being responses in arithmetic which are purely learned. Even when first learning has been satisfactorily accomplished and a period of vacation is introduced without opportunity for frequent use of learned combinations, skills, and the like, maturation is unable to produce any increase in ability or even to hold the ground already gained. Pupils compute with less facility in September than in the previous June, in spite of three months of physical growth. Children who have been drilled well in June withstand the vacation period with much less loss than those who have not had such drill.

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[3] Buswell, G. T. and Judd, C. H., *Summary of Educational Investigation Relating to Arithmetic*, pp. 103, 104. University of Chicago Press, 1925.

[4] Wheeler, Raymond Holder and Perkins, Francis Theodore, *Principles of Mental Development*, Chap. XIII. Thomas Y. Crowell Company, 1932.

The drills which Kirby, Thorndike, and Hann used in their experiments were all of the type referred to throughout this chapter as *isolated drill.* Since the outcomes of their experiments showed unquestionably that pupils who were drilled in arithmetic were more efficient than pupils who received regular instruction without drill, the proponents of drill in arithmetic went to work with renewed vigor, apparently quite careless of the question about the effectiveness of different types of drill organization. The only change in the drill offered seemed to be a slight difference in the length and frequency of the drills. Drill sheets outside the textbooks came into some favor but maintained most of the old features. These practices gave clear indication that insight into types and purposes of drills was somewhat lacking.

Purpose of drill. Drills, like other teaching devices, have only one main purpose, which is to increase the pupil's understanding and facility in the use of desirable facts, skills, and information. To get the most good out of the time and energy expended in exercise we must construct or select the type which will serve our purposes best. There are two types of drill organization now in use, *isolated* and *mixed.* In the isolated type intensive exercise of a few skills is emphasized. In Table 1, which is an illustration of

TABLE 1

THE ISOLATED TYPE OF DRILL ORGANIZATION

1. $2/9+5/9=$	2. $1/2+1/2=$	3. $5/7+2/9=$	4. $2/3+3/8=$
5. $9/10+1/5=$	6. $5/9+3/7=$	7. $8/9+1/6=$	8. $4/5+1/2=$
9. $2/3+5/6=$	10. $3/5+7/8=$	11. $3/5+2/7=$	12. $8/9+1/9=$
13. $7/8+2/3=$	14. $3/4+5/6=$	15. $3/5+1/2=$	16. $8/9+7/8=$
17. $1/2+1/4+1/3=$	18. $3/4+5/8+1/2=$	19. $5/7+2/9+3/5=$	20. $1/8+1/7+1/4=$

the isolated drill, it will be noted that all problems are concerned with one process in arithmetic.

A brief survey of Table 1 reveals that this is the type of exercise which has appeared in arithmetic for years, and its chief use was confined to reviews given at the end of each term, semester, or year. That is to say, the older, and some of the more recent, authors of arithmetic textbooks have advocated, by their textbook productions, the use of isolated drill for maintenance or review purposes. This, as will be pointed out later, is a practice which is not wholly defensible in light of recent experimental evidence. The fact is that isolated drill is now increasingly used for a very different purpose. Further reference to Table 1 and the inherent characteristics of isolated drill organization will be made presently. The second type of drill, shown in Table 2, is the *mixed* drill organization, in which many skills and processes are included in one drill exercise.

Until recently, mixed drill was not to be found in any of the

TABLE 2

THE MIXED TYPE OF DRILL ORGANIZATION

1. $14\overline{)8599}$

2. $\begin{array}{r} 6\,{}^{1}/_{2} \\ \underline{-\ {}^{1}/_{3}} \end{array}$

3. $27\overline{)538}$

4. $\begin{array}{r} 8963 \\ \underline{\times\ 38} \end{array}$

5. ${}^{1}/_{4} \times 12 \times 5 =$

6. Subtract:
$9\,{}^{1}/_{2}$ from $10\,{}^{4}/_{5}$

7. $\begin{array}{r} 80856 \\ \underline{-77184} \end{array}$

8. $\begin{array}{r} 976965 \\ 9766 \\ 85 \\ \underline{27234378} \end{array}$

9. Multiply:
438 by 577

10. $\begin{array}{r} 295 \\ \underline{\times 389} \end{array}$

11. $4\,{}^{2}/_{3} + 7\,{}^{1}/_{2} =$

12. ${}^{3}/_{10} + {}^{2}/_{5} =$

13. $\begin{array}{r} 84 \\ 31 \\ 99 \\ 46 \\ 77 \\ 665 \\ 3477 \\ 89 \\ \underline{68} \end{array}$

14. ${}^{3}/_{4} \times 2 \times {}^{1}/_{3} =$

15. $65\overline{)1823}$

16. Add:
$\begin{array}{r} 998 \\ 239 \\ 234 \\ 910 \\ \underline{629} \end{array}$

17. $\begin{array}{r} 9912 \\ \underline{-7383} \end{array}$

18. Subtract 4 bu. 2 pk. 1 qt. from 8 bushels.
Answer.............

19. Add:
$94\,{}^{1}/_{2}$, ${}^{2}/_{3}$, 26, $10\,{}^{3}/_{4}$

20. Multiply:
$\begin{array}{r} 4209 \\ \underline{63} \end{array}$

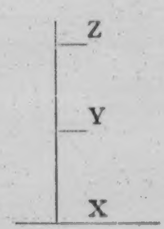
textbooks or courses of study in arithmetic with which the writer is acquainted. However, due to the efforts of F. B. Knight, G. M. Ruch, and others, this type of drill is rapidly increasing in favor and is displacing the isolated drill for some purposes. While drill has been and is still open to attack on many fronts, it continues to be used and when the program is properly organized, not only is the arithmetical ability of pupils increased but this condition is also accompanied by a perceptible degree of satisfaction on the part of the learners.

As indicated in a preceding paragraph, there is not unanimous agreement that drill, as the term is commonly used, is essential or even desirable. Some proponents of Gestalt psychology would discard such exercises altogether. Wheeler and Perkins are quite specific in their statements concerning the law of exercise. They say that learning is a continuous process of making discoveries; that it involves doing something that was never done before. It is growth in a specific situation—anything but a repetition of a performance. This apparently discredits the use of drill altogether. Nevertheless, after careful study of the entire proposal of a learning program by Wheeler and Perkins, one is impressed with the soundness of well-constructed drill as a means of increasing the effectiveness of teaching in almost any situation. The drill proposed by Gestalt psychologists, if it is to be given consideration, will need to be modified in a number of ways. One of the chief defects of exercise materials, as they are now set up, is the sameness of the situations, that is, the pupil is tempted, if not forced, to repeat responses because the drill situations are identical with situations to which they have already responded. Now the ideal drill on every desirable fact, skill, or ability is such that each situation is a new one; for example, $2 + 2 = 4$ would be presented in equation form one time, in column form another, in a problem setting another, and so on. Obviously there are limitations to the amount of variations that can be employed in the outward setting of the drill situation. Since the total situation includes the pupil factor and numerous other factors besides the drill sheet, the exercise is never the same and thus responses cannot be merely repetitions. Still, marked dependence upon variations in the experience of the pupil and in the total setting of the exercise period, other than the drill sheets or exercises, is in no way justified. The point made by the Gestalt psychologists, while not a new idea, is vital. We must, if drill is to be effective, present situations in such fashion that the pupil is constantly alert and not quiescent, that he constantly learns old facts a little better, that he learns how to apply them more broadly, and that he learns some new facts as well. Good arithmetic drill exercises have always considered these features. Thorndike and his followers have long advocated and supported such concepts.

The chief place which isolated drill should occupy in the learning process is in close proximity to the first teaching and learning of new facts, skills, and information. It should be noted that this type of organization lends itself admirably to such use. That conditioning of the organism resulting from first experience with such a fact as $\begin{array}{r}6\\+2\\\hline 8\end{array}$ is slight, we all agree. The depth of the impression made on the nervous system of the child is usually not great, say, like U. What everyone concerned desires is an impression which is deep, clear-cut, and lasting, say like U. One effective way to bring about this desired condition is by intensive, interesting, and concentrated exercise just following first teaching and learning. It is significant that practically all the better textbooks in arithmetic now provide such exercises. The trend is definitely toward the use of isolated drill in connection with first teaching. The "stamping in" process can hardly be accomplished without it. Much of the isolated drill will be anticipatory in nature. For example, preceding multiplication a considerable amount of drill in certain higher-decade addition, such as $45 + 3 =$, $28 + 4 =$, and $36 + 7 =$, will be decidedly effective. The teacher of arithmetic can doubtless use much more material of this nature than is now available. There is no good reason why, until this lack is corrected, teachers should not develop their own isolated drills for such purposes.

When properly used, isolated drill works in helpful fashion and in complete harmony with two important laws of learning, that is, exercise and effect. No pupil of reasonable ability fails to profit significantly by exercise of responses recently acquired, with pleasure, for to exercise a response pattern newly established always gives satisfaction under normal school or home conditions. Concentrated drill upon a few of the newest responses in arithmetic through isolated types of exercises in the form of work sheets tends to strengthen these responses and to decrease possibilities for confusion. Such errors as $7 + 2 = 11$ and $4 + 7 = 9$ may easily occur unless each of these and other similar facts receive sufficient exercise in isolation, just after first learning and before meeting them in company with other basic combinations.

One of the chief questions to be decided upon relative to the use of isolated drill for fixation of first learning is: What degree of overlearning at the time of first learning shall be accomplished? Under our American system of mass education we must, to a degree, teach in groups. Obviously a given amount of teaching effort, drill material, corrective work, and the like, will produce different amounts of overlearning in pupils taught in the same class. Consider the figure at the left. On the whole, should we strive in first teaching and drill to reach only level *X* which is merely the ability to respond once correctly after first learning, including isolated drill, has been completed? Should this pupil be encouraged to reach the level of overlearning represented by *Y*, or should the average pupils in a class be expected to respond with a degree of precision, accuracy, and ease represented at *Z*, which is somewhere in the neighborhood of the physiological limit? Obviously, to attain level *Z* will require much more time and effort than that required for the achievement of level *X* or level *Y*. Since there are other worth-while activities to be engaged in beside arithmetic, a limit must be placed upon the amount of time and energy to be devoted to the learning of this subject. Furthermore, every teaching program must take into account the inevitable forgetting curve. True, we tend to remember more effectively the nearer we mount to the physiological limit, but whether the pupils overlearn much or little, nature tends to erase the impressions, responses, or neural scars, and often with alarming rapidity. It seems defensible to assume that we should encourage the pupil in his first learning efforts to reach, at least, level *Y*. Such a schedule of first-learning effort will, through wise use of time saved, permit some much needed attention to be given to the matter of *fighting forgetting*.



Maintenance of skills and abilities. We now turn to a consideration of the matter of maintenance of skills and abilities after first teaching and learning have been successfully accomplished. We can hardly limit ourselves to two laws of learning, exercise and effect, if we have proper insight into the learning habits of children. We must, in considering the drill program, break the laws of exercise and effect into several component concepts, such as the law of use, the law of frequency, the law of recency, the law of disuse, and the law of effect. It is not the intention here to recite the laws of learning with which every student of education or psychology has become so familiar that they have in some cases degenerated into a "lingo." The purpose is, however, to take these laws which are the very backbone of any good method of instruction and apply them to the construction of a well-integrated drill program.

The facts relative to forgetting force us to give some attention to systematic review of crucial knowledge and skill in arithmetic. The farther the pupil progresses in arithmetic, the more he has to remember and the more complex the matter of review and maintenance of ability becomes. This is apparent when we consider the number of facts and skills that a sixth grade pupil must remember with facility as compared with the number that the same pupil was required to know and use as a third grade pupil. At the end of the first half-year in the ordinary third grade a pupil is usually required to know many of the facts and skills involved in the addition and subtraction of whole numbers, a limited number of facts and skills in multiplication, a still more limited number concerning division of whole numbers, and very little about common fractions and denominate numbers. The sixth grade work in the same school requires the pupil to do accurate and fairly rapid work with all the processes in whole numbers and fractions and, in addition, to be able to work with decimals and denominate numbers. Where a pupil in the third grade must maintain, let us say, a thousand facts and skills, the pupils in the sixth grade must maintain many times that number.

There are two methods of constructing drill for maintenance purposes in this grade or in any grade. One is by means of isolated drill construction, which is shown in Table 1. By this method, addition of whole numbers, which is only one of twelve or fifteen processes to be used by the average sixth grade pupil, would receive approximately twenty minutes of review (drill) in isolation about every twelve weeks. Every other process would have its turn in the isolated drill program about every twelfth week. The relative inadequacy of such a schedule of drill can be illustrated by one typical case. For example, addition of whole numbers would receive twenty to thirty practices the first week of the term, twenty to thirty the twenty-fourth week, and twenty to thirty the thirty-sixth week. During the remainder of the school year no direct attention would be given to addition of whole numbers. Other skills would be treated in similar fashion. The rise and fall of the learning curve under such conditions would appear as in Figure I. While gain is significant, considering the whole school year, there is too much opportunity for forgetting between "bunches" of isolated drill on the various topics. The difficulty is that the laws of recency, frequency, and disuse are largely neglected. The periods between drills are almost as long as the average summer vacations. Nearly all drill ignores the law of effect, for to forget something which one wishes to remember is keenly annoying. Failure to remember crucial facts may be so acutely unpleasant that the pupil will develop a definite negative mental set toward all arithmetic work.

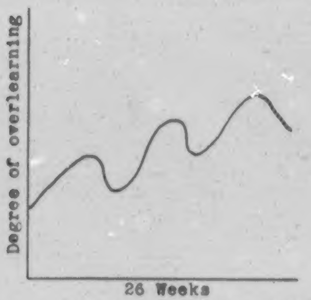


FIG. I. The effect of "bunched" or isolated drill which causes the rise and fall in the learning curve. Half of the time it reverts to a forgetting curve.

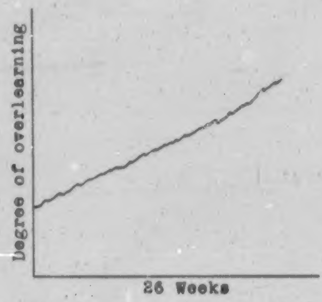


FIG. II. The consistent gain in mastery of processes receiving small amounts of exercise weekly.

Another and a much better type of drill organization for maintenance purposes is mixed drill, such as that provided by an exercise. See Table 2. When the same examples that were used in making the isolated-type series of drills are organized in mixed fashion, the distribution in amounts and time is appreciably better.

In such a program almost every one of the twelve or more processes dealt with receives some practice every week. We still have approximately twenty to thirty problems for a twenty-minute drill period, but exercise is given on many skills every week.

Figure II shows how the curve of learning progresses when a good mixed-drill schedule is put in operation following good first teaching in which is included valid isolated drill. Every skill received on the average two exercises each week.

Here it will be noted that the learning curve rises gradually but regularly. That is to say, the maintenance exercises of the mixed type preclude significant forgetting and actually force the level of overlearning to rise in consistent fashion. In such a program every law of learning—use, disuse, frequency, recency, and effect—is adequately considered.

An experiment[5] in which the variable was distribution of practice was conducted in 1928 with several hundred school children divided into two equated groups. While both groups gained as a result of the twenty-six-week drill program, the group which was permitted to exercise skills often and to exercise each skill in small amounts gained 23% more in accuracy of response than did the pupils subjected to exercises of the isolated-drill type. A further significant fact was revealed when each group of pupils was divided into three parts as to arithmetical ability, and compared. The lowest third in the group using mixed drill gained 54% more in arithmetical efficiency than pupils of equal initial ability who had used isolated drill for the same length of time (26 weeks). The middle third of the group using mixed drill surpassed the corresponding third using isolated drill 25% in gains, and the upper third using mixed drill showed 8% better gains than did their paired competitors using isolated drill. In 1933 an experiment with like groups was again conducted by the writer. Results favored again the use of mixed drill for maintenance purposes. The 1933 experiment was not so well controlled as the 1928 experiment and for that reason could not be considered as reliable. The writer, in 1934, conducted a third experiment of like nature with one hundred and ten college students. The data did not significantly favor mixed drill for maintenance purposes.

A unified drill program. A useful concept to keep in mind in connection with the use of mixed and isolated drill may be set forth as follows: Such a program of drill will make a place for both types of drill organization and each can be used to good advantage. Each skill is best impressed into the pupil's nervous system at the beginning by use of carefully constructed isolated drill. From that time on through the use of mixed drills no skill is allowed to go long without exercise. The fact that each problem is different from the next in the mixed drills tends to overcome any tendency toward boredom. Furthermore, the degree of success is likely to be consistent and reasonably high on each week's drill work since there are no whole pages of problems such as are found in long division or percentage reviews, every one of which is difficult for the class or some particular pupil. While some problems are likely to be difficult, in mixed drills others will be easy, thus offsetting the discouragement arising from the undue difficulty of a few. The mental set, as a rule, constantly grows in a positive direction when isolated drills are used for fixation purposes and when mixed drills are used for maintenance purposes. A unified drill program of isolated and mixed drills will require no more time than a program using but one type of drill, if review comes, as it should, about once a week for maintenance purposes. There need be no such thing as reviews on addition one week and on subtraction the next. In contrast, a varied diet of reviews would be the regular experience week in and week out.

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[5] Repp, Austin, *Twenty-ninth Yearbook of the National Society for the Study of Education*, Part II, Chap. VI. Public School Publishing Co., 1930.

While isolated drill will be used most frequently to impress behavior patterns newly acquired, this type of drill organization has a second important place. Following the discovery of weakness in any particular phase of a gross skill through the use of tests, mixed drills, or other means, isolated drill will prove valuable. Here we have a case quite similar to a first-teaching situation. In some respects it is more difficult than a first-teaching situation because wrong response patterns must be broken down before correct responses can be established. In any case, exercise of the type afforded by use of isolated drill will be most helpful.

Results of continued experimentation, together with some practical considerations growing out of the use of the two general types of drill construction, tend to make one cautious about any sweeping generalizations concerning one type of drill for one purpose exclusively and another type for another purpose. While the positions stated in the preceding paragraphs seem to be tenable, the writer is inclined to believe that mixed drill has some very definite limitations even for maintenance purposes after a certain level of achievement has been reached in the mastery of fundamental arithmetic facts. There are a number of reasons why this is so. In the first place, forgetting does not take place as rapidly and as completely as we have sometimes thought. Smith[6] in a recent study with about one hundred and fifty fifth grade pupils found, as have others in recent studies, that the loss of factual information in arithmetic during a ninety-day summer vacation was not extreme. This would seem to indicate that review on various skills well mastered, say up to normal mastery at the sixth grade level, need not come in small amounts weekly but might well come at intervals a little more widely spaced and in relatively larger amounts. The writer's experience with college students would lend some support to this concept. Myers and Myers' study also lends support to this position.[7] In the second place, every drill is really a test or examination and should have diagnostic and remedial possibilities. Diagnosis is so sketchy and so widely spread as the result of an extreme mixed type of drill that its usefulness in the way of remedial work is limited. To the writer it would seem best for the teacher to use a modified type of mixed drill beyond the fourth grade, if best results are to be achieved. By modified mixed drill is meant exercises in which about four processes will be exercised five or six times in a twenty-minute drill period. This will not be the old isolated drill; neither will it be exaggerated mixed drill. Table 3 is a good example of modified mixed drill, in which many skills and processes are exercised each week.

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[6] Smith, Lulu Forrest, "A Comparative Study of the Persistence and the Recall of Learning." Master's thesis, University of Arizona, Tucson, 1934.

Method of presentation of drill. It is a commonly known fact that drill with respect to method of presentation and response may be any one of three types: oral, written, or oral and written. Which type is best for teaching and learning purposes has not been definitely established. In order to learn something about this question the writer in April and May of 1934 directed an experiment involving a small number of third grade pupils over a period of six weeks. The three groups were at the outset equated as to arithmetic ability. Each contained pupils of low, average, and superior ability. All three groups were taught identical, new material in arithmetic. Following first teaching Group I was given oral drill, Group II written drill, and Group III oral and written drill. Such factors as teachers, amount of drill, and drill time were kept constant for all groups. The teaching was equated by using Teacher A for Group I two weeks; then she was shifted to Group II, and then to Group III. Teachers B and C were alternated in like manner. This was possible because there was available a good supply of student-teachers who

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[7] Myers, Gary C. and Caroline E., "The Cost of Quick Shifting in Number Learning." *Educational Research Bulletin* (Ohio State University), 7:327-334, October 31, 1928.

TABLE 3

MODIFIED MIXED DRILL IN WHICH DIAGNOSIS AND REMEDIAL WORK ARE FACILITATED

1. $497 \times 685 =$
2. $50.75 \times 86 =$
3. $100 \times 3.85 =$
4. $250 \times 40 =$
5. $79086 \div 98 =$
6. $39\overline{)11986}$
7. $40\overline{)387800}$
8. $9\underline{)82761}$
9. Add 39, 29, 25, 14, 173, 86, 99.
10. $404 + 91 + 80 + 375 + 166 + 384 + 162 + 198 =$

11.
5
9
8
4
6
7
5
0
$\underline{3}$

12.
17
38
7
309
64
87
46
$\underline{152}$

13. $8\frac{1}{2} + 4\frac{3}{4} + \frac{7}{8} =$
14. $\frac{2}{3} + 1 + 3\frac{1}{2} =$
15. Which is greater, $\frac{2}{3} \times 40$ or $39 + 1\frac{1}{2}$
16.
$12\frac{1}{4}$
$3\frac{2}{3}$
8
$\underline{1\frac{5}{7}}$
17. $5\frac{1}{2} \times 2 =$
18. $\frac{3}{4} \times 1\frac{1}{2} \times \frac{5}{6} =$
19. $27\frac{1}{4} \times 2\frac{1}{4} =$
20. $0 \times 2\frac{1}{2} =$
21. Subtract 4 pk. from 6 bu.
22. How many sq. rd. in 5 acres?
23. Which is larger a field 80 rd. square or one containing 25 acres?
24. Divide 8 bushels of apples into 5 equal portions. How much is in each?

were completing a four-year teacher-training course in college. At the end of six weeks it was found, by use of a combined oral and written test, that Group II had profited most as a result of the work; Group III achieved the next best results; and Group I made the least gain. This would indicate that written drill following first teaching in arithmetic is superior to either of the other types. Motivation was achieved through such devices as charts and class work, and through encouraging and commendatory remarks by teachers, as well as by other means of recognition. Perhaps the most important points in favor of written drill as opposed to oral are: each pupil works at his own task undisturbed by others, there is little waste of time, stimuli are relatively stable instead of transitory, undue emotional strain is avoided, the drill can be, and usually is, well planned instead of extemporaneous. Competition may be less keen in written than in oral work, but this negative factor seems to be offset by others of a positive nature. While the number of pupils in these groups was too limited (27) to yield conclusive evidence the experiment was carefully controlled and the results were consistent.

Since it will be impossible in most cases for the classroom teacher to construct for pupils all the drill that is necessary in a well-integrated system of learning, it will often be necessary to select rather than to construct the drills. At present the tendency is definitely in the direction of a dual but unified drill program of the type suggested in this chapter. The danger to the pupil lies in the fact that many of those who construct drills give only lip service to the laws of learning. Certain factors need to be carefully checked in selecting the exercises to be used. First, difficulty and cruciality of the facts and skills included in the practice exercises need to be studied. Often we find that easy processes and combinations are exercised frequently, while in comparison difficult combinations and processes are greatly slighted. Furthermore, those facts and processes which are more crucial in *adult life* need to be included frequently in drills, and unimportant processes should, in comparison, be neglected or even ignored. If a drill is to be effective it must not only bring about exercise but highly motivated exercise as well. Several factors will aid in bringing about increased motivation. Items should be arranged somewhat in order of difficulty. That is, as a general rule the easier items should come first and the harder items should follow in order of increasing difficulty. On this point of arrangement of items agreement seems to be unanimous among recognized authorities on test construction. The writer would like to suggest that this rule may not be wholly valid. It may very well be that an easy item placed well toward the end of a drill along with more difficult items in the exercise will aid the pupil to make a new start and achieve more in the whole drill than would otherwise be the case. Experimentation on this point would be exceedingly interesting and perhaps valuable. The drill sheets should be attractive in appearance. This may be assured through the use of printed characters of proper size, good quality of paper, and pleasing mechanical set-up of the sheet. Drills which have carefully determined standards of achievement are on the whole more conducive to continued pupil interest than those without such standards. However, Panlasigui[8] shows definitely that failure in arithmetic may be made so keenly annoying to children in the lower brackets of a class that frequent attention to standards not only may be disturbing, but may actually retard progress which would otherwise be within the pupil's capacity. For dull children or for pupils who make low scores standardized drill sheets seem to be less useful than those without standards.

Generally speaking, the proper use of drill in arithmetic needs to be better understood. Correct drill construction and its use is still a fertile field for research. The teachers of arithmetic who more carefully and effectively divide the time between first teaching, fixation, and maintenance will be repaid a hundred-fold in increased mastery by their pupils. Perhaps the recent attempts at drill organization have swung a little too far to the left. We may need to return somewhat to the right.

While this article is concerned primarily with the use of drill in the teaching of arithmetic, it will not be out of place to say that whatever truth has been discovered about types and effectiveness of drill organization in arithmetic might well be transferred and used in the teaching of every other school subject.

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[8] Panlasigui, Isidoro and Knight, F. B., in *Twenty-ninth Yearbook of the National Society for the Study of Education*, Part II, Chap. XI. Public School Publishing Co., 1930.

RETROSPECT, INTROSPECT, PROSPECT

By DAVID EUGENE SMITH

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This essay is written after a careful examination of the preceding articles in the book. These articles show a degree of scientific research which all readers will commend and which, it is safe to say, will assist in the training of future teachers of arithmetic in particular and of mathematics in general.

This article, begun with an array of facts which admitted of statistical treatment, has turned to another path, one leading to a field of combat. It begins with a brief historical summary—a Retrospect. It passes to a moment of introspection, coining a word merely for the sake of euphony—Introspect. It then moves on to the dangerous field of prophecy and combat—the Prospect.

RETROSPECT

The field of Retrospect is that in which the pessimist has always been at his best. To him the world has always been going to the dogs. Not so many years ago a very few radicals ventured to remark that most of the arithmetic then taught was absolutely useless for the vast majority of pupils. The result was a turmoil interesting to watch, just as the present article will lead to combat. One example held up before bodies of teachers was General Average—a term long since useless, but familiar to teachers of arithmetic some time after I began their training. To the student of social activities in the nineteenth century, or of the influence of geography upon these activities and upon arithmetic, the study of the meaning of this topic today is interesting. For the schools of this country as a whole it lost a century ago whatever of value it had.

In the same way the last generation or so has seen "thrown into the discard" such monstrosities, from the standpoint of the real needs of the vast majority of people, as equation of payments, partnership involving time or partnership problems of any type, partial payments, compound proportion, the learning of number names to vigintillions, duodecimals, continued fractions, repeating decimals, apothecaries' tables together with Troy weight, progressions, the writing of Roman numerals beyond any needs of the present, cube root, and (of late) square root, and the unused and unusable parts of the metric system. These are some, but not all, of the subjects taught to teachers in our training schools a half century ago—taught to be retaught in "the little red schoolhouses" all over the country. Pessimists said that we could not improve upon this offering, that we had nothing with which to replace it, and that it was good for the soul if for nothing else. That the demand has not ceased for traditional material which is practically obsolete, is evidenced by a passage in Professor Morton's chapter. He states that, in connection with curriculum making, certain teachers suggested the inclusion of partial payments and paper-hanging. In certain localities these have some value, but in a course for general use in this country, they have no place. In the same study it is interesting to observe that 59 per cent of those voting on the curriculum asked for more time to be given to common fractions.

Here, then, are a few concrete cases of definite progress in a single branch of school activity. This progress came gradually, but it came surely, as a result of the common sense of, at first, a relatively small number of teachers and doubtless through the help of the parents of the children who had to be mentally tortured by the demands that these subjects made. See, however, what this change meant. Out of arithmetic was taken about half of the material, and the pessimists saw in this mutilation the destruction of one of their cherished pillars of education.

So we see that we have cause for optimism today. If such a remarkable change could be made in arithmetic in a half century, a period in which conservatism was in power and when speed in matters of educational reform was frowned upon by most of the profession, why should we not expect a still greater advance in the period before us?

All this has manifestly no bearing upon the methods of teaching primary arithmetic—the arithmetic of number facts. This has been discussed in a very scholarly way by Professor Brownell in his monograph in this book. In his study he, too, has looked at the past and has suggested the possibility of improvement in the future, not in the subjects taught but in the way in which the skill in computation is developed. It is manifestly necessary that the teacher should consider carefully both of these problems in attempting to improve the presentation of the subject.

INTROSPECT

And now, what of the present? Why not indulge in a brief introspection which may help us to view the future with more profit and with more optimism?

Confining ourselves to the mathematics—if we must use that word in this sense—of childhood, let us see what elementary arithmetic is offering today. Looking superficially over this field we see the numerals which the Arabs never used but to which we have given their name. We also see the numbers and forms which the Romans scattered over their wide domain. If Caesar could see them as they are taught today, he would need to go to school to see what some of them mean. Does this introspection give us any idea of change for the better in this narrow region? Let us also look at our number names—are they like the laws of the Medes and Persians, immutable to those who made them, but dead to the world of today? Ask any scholarly person in Great Britain what a "billion" was before the World War. Ask him what it is tending to become today. Ask yourselves what you think it has always been. Ask the scientist what it will be tomorrow.

We have "four fundamental operations" in arithmetic and in algebra as well. Are these forever fixed? At various periods in relatively modern times there have been five, at other periods six, and so on, up to nine. Moreover there are many who would agree that there is only one operation—that of addition. We may say that all this is a useless quibble over words, as indeed it may be, but are all of these or other operations of equal value and are they to be taught as such? If so, must we still cling to the phraseology which comes to us, chiefly from the sixteenth and seventeenth centuries when Latin was the language of the scholar? Indeed, why do we need the term at all? As to details, many a reader of this article, if many there be, has to stop and think which is the minuend and which the subtrahend when he subtracts. When he says, "*Deduct* what I have paid and I will give you the *rest*," he uses the language of the business world. If anyone should say, "*Subtract* what I paid and I will give you the *remainder* or the *difference*," he would be a subject of derision—a pedantic way of saying that he would be laughed at.

Think of the trouble that the teacher has in getting a child to remember which is the "product" and which is the "sum." The child sees no sense in the distinction, and he is right. It is only a short time ago that "product" was also used for the result in addition and in division, and even today when we write "carried forward" or "brought forward," carrying a sum to the next page in an account book, we are similarly translating "product" (carried forward) into another form.

This is only one of dozens of instances in which our language of the schoolroom is not the language of trade or of daily life. You may say, "Well, what's the *odds?*" The question is legitimate, but when you ask it you are using an old mathematical term, one which is now forgotten.

Even the operations themselves will, in careful scrutiny, appear to be open to much valid criticism. It is easy to say that the quotient (what a word with which to bother children!) must be placed above the *numerus dividendus* (which we keep as a relic in the form of "dividend"), but by so saying we get into a mess of trouble. In the first place the word "dividend" is generally used with an entirely different meaning outside the schoolhouse from that which is taught so uselessly and with some little difficulty in the classroom. In the next place, the argument for placing the quotient above in short division, as well as in long, is open to a very dangerous attack. The real crux of the matter comes down to this: Why do we encourage short division anyway when for most people it is longer than the other?

This matter of using unnecessarily difficult names is summed up very succinctly by Professor Brownell in his contribution to this book: "Arithmetic is singularly unfortunate in the language which has come to be used to describe the processes by which its subject matter is to be learned and to be taught." In speaking of the teaching process, he is referring to the language of the "educator" and he shows that the same stupidity is shown in the field of educational theory as was and still appears in the presentation of the first elements of arithmetic. In such a brief sketch as this it is impossible to carry this phase of our introspection much farther. Suffice it to ask, do we not feel that possibly the language of arithmetic today is open to wholesome change? Must we always have addends, sums, differences, remainders, multiplicands, products, factors, least common multiple (which to the child is most uncommon), greatest common devisor, numerator, denominator, base, rate, percentage, partial products, and various other relics of the past? It is no very satisfactory answer to say, "But we *must* have them," or even, "Well, what would you use in their place?" The assertion is not true, and the question is easily answered by anyone who has given the matter thought. The terms were originally introduced to be memorized along with rules, a method of teaching now discarded.

But leaving the vocabulary out of the question, are we always going to add long columns of figures in the classroom? Ask the banker with his adding machines or the merchant with his cash register—now to be found throughout the civilized world. Must we always haggle over the "addition method" or the "taking-away plan in subtraction" or should we consider what the world is now doing, and what it ought to do for convenience? Is it more convenient to write a result at the top or at the bottom of a computation, when it must often be multiplied, divided, or added? Must we try year after year to teach all children to do long division with decimal fractions, except in simple cases with dollars and cents? Would it not be a good plan for us to carry our introspection far enough to ask when any one of us ever had occasion, outside the schoolroom, to divide, say 203.047 by 4.2693, or whether we have ever heard of anyone doing it by any schoolbook method? Of course, the answer is that such cases are no longer found in our schools, but what about dividing 34.2 by 1.09? This is found, and so in fact is the other. Is it any answer to say that such cases are needed in the laboratories of science when it can be taught there in five minutes if necessary?

Coming to a point where combat is more sure to arise, what about the common fractions which are so uncommon if not happily obsolete? Look at the monstrosities found in certain of our "tests" today. The pun on "detest" would be almost allowable in these absolutely unusable cases. Fractions such as have not been used in practical work for two centuries are given under the pretense that they are useful in teaching a "process." Are we prepared to say that a process, which in any case is of doubtful value, is to be taught by the use of difficult fractions which no sane person would ever attempt to use in daily life or even in the complicated formulas of the laboratory?

But this introspection should not always lead to the pretended discovery that the arithmetic of today is all bad. On the contrary, the American textbooks in arithmetic today are, for the purpose of giving what most pupils will need in the future, the best in the world. I say this with perfect confidence, for it has been my duty and pleasure during many years to know the best books of all the leading countries. In one respect Europe excels us, namely in a certain form of thoroughness, but in spite of this, our textbooks are far in the lead, for our purposes, considering the democracy of our education. When we come to a semi-aristocracy of learning, separating those who early show a taste for mathematics from those who have no such aptitude, our problem will naturally be modified, but not in the direction of retaining that which is useless.

That our textbooks are not perfect is apparent to anyone who examines them even superficially. If the reader should apply introspection with a broader motive than that of evaluating the subject matter, he will do well to read Professor Buswell's chapter in this book. The general nature of social arithmetic is there set forth with the author's well-known ability to state his propositions clearly. Especially important is his discussion of the probable values of the material offered in the seventh and eighth grades—a problem which every textbook writer has to meet, and one which changes from decade to decade.

PROSPECT

What is the outlook for the future? If we have such good textbooks now, why should we try to improve upon them? Even if our curricula are not very good, are they not good enough? Of course the question would not be asked seriously by anyone gifted with ordinary intelligence, so let us see what the future may have to offer, not that the offering need be accepted but that it is deserving of consideration.

First, as to the vocabulary. The early schools, influenced in Europe by the Latin tongue, said, "This is a number to be added (*numerus addendus*)," a phrase perfectly clear to the Latin-taught pupils. We took out all the meaning for children when we cut this down to "addend." We do not use the term often enough to justify it in the schools, and we do not use it at all in daily life. Why not say, "Let us add these numbers"? Furthermore, why do we need to perpetuate any of the list of terms already mentioned above? It is perfectly easy to say that this is the number to be divided, just as the early writers said it in Latin. We have forgotten the Latin meaning of "dividend," but have kept it in school with a meaning which, as already stated, is entirely different from that used in business.

As to the numerals, there is no apparent need for a world change. It is evident to everyone who has given any thought to the matter that, mathematically and commercially speaking, it would be better if the race had developed twelve fingers instead of ten, which would have given us a scale of 12. We recognize its value in our 12 inches, 12 ounces in the old pound, and 12 shillings in the British monetary system. To make a world change from a decimal to a duodecimal system is, of course, out of the range of possibility, unless and until we have a world dictatorship, say a million years hence.

When we consider the time wasted on the Roman numerals, however, the future will probably allow these to be taught only far enough for reading dates and chapter numbers. There is not the slightest need in present-day business for reading or writing numbers between CC and MDCC, and very little for any beyond 1900. The latter we commonly see written MCM, a form which no Roman would ever have used, and whose meaning but few would have guessed.

As to operations, there will always be, so far as can be seen at present, a need for adding short columns of figures in the grocer's bill—not by the grocer, for he will use an adding machine (cash register), but by the purchaser who adds them in his account book. Similarly, the outlook seems to show that in the next half century all the operations will generally be done mechanically except for a few ordinary computations of little difficulty. Even today the slide rule is so inexpensive as to be used by thousands of workmen in shops. We may reasonably look to the time, not far off, when an inexpensive computing machine, the size of a watch, will be in general use. We teach the use of the typewriter in hundreds and probably thousands of schools today and they are even entering the elementary grades, and the prospects seem to be that the future will see the teaching of the calculating machine in the same way. The day of the "lightning calculator" is, for practical purposes, past; and the time for adding columns of figures longer than needed for ordinary bills is passing.

As to fractions, the children in our schools today need the halves, thirds, and fourths; they have some use for fifths and eighths in the daily life of most people. This is recognized in Dr. Buckingham's paper on Informational Arithmetic. Other denominators like tenths, twelfths, and sixteenths, will doubtless continue to be taught, but the words "numerator" and "denominator" will probably be replaced by shorter terms and die a peaceful death, unmourned and soon forgotten. The meaning of the symbols L. C. D. and G. C. D. will also go, since no operations with fractions, even in scientific laboratories, will need them. In fact, they have not been really necessary in business or in scientific work for many years. The reducing of fractions to lowest terms, generally useless since the invention of decimals, will be among the historical curiosities, and its handmaid factoring, except in the simplest cases, will be put by its side in the museum. What Professor Overman, in his very helpful essay on the problem of transfer, has to say about the operations on the numerator and denominator of a fraction today may very likely be read twenty-five years from now as an interesting study of a forgotten topic. Even at the present time the teacher who may ask what we are supposed to do with a fraction like $^{15}/_{75}$ should understand that fractions of this kind are obsolete. The fractions which the pupil or his parents will use at the present time involve little, if any, reductions to lowest terms, and these will rarely extend beyond such cases as $^{4}/_{8}$ and $^{12}/_{16}$.

But above all, we should bear in mind that it is not today that concerns us, it is the future. Neither is it the offering of textbooks; this is properly conditioned by the work of the curriculum makers, and it is upon their judgment the nature of the arithmetics of the next quarter of a century depends. Always the question is not what pupils can do in any particular grade, but what they should be asked to do. It is for this reason that the worst way of constructing a curriculum is to base it on what teachers say that the pupils in their several grades are able to accomplish. They are able to become proficient in writing numerals in Chinese, say in Grade 3, but this is no reason for saying that this accomplishment should feature there or anywhere else in the elementary school.

The work with decimals, for all except those who have special interests in number work, will probably be limited to dollars and cents, and to decimals rarely exceeding two or three places. Such unreal monstrosities as the addition of "ragged decimals" will soon go. Teachers will realize the significance of "degree of accuracy" in measurements, and that in any given problem this degree will be constant. It is interesting to know that the value of *pi* was long ago computed to upwards of seven hundred decimal places, but this has no practical value whatsoever, and the use of any approximation beyond 3.14 in our elementary schools in connection with finding the circumference of a circle, of which measurement shows the diameter to be 3.27 inches, would show mere stupidity. In practical work in mechanics all computations relating to decimals are already done with tables or with slide rules or other more elaborate machines, and these are the methods of the future. Professor Johnson's paper on "Economy in Teaching Arithmetic" has some interesting data showing that "common" fractions have always occupied more space than decimals in the textbooks, and therefore that the latter are much more easily taught. This is seen even more clearly when we consider that decimals, as used in dollars and cents, are much more important in the daily life of our people.

In the illuminating study entitled "Opportunities for the Use of Arithmetic in an Activity Program," it is significant that the table showing the totals of uses of integers, fractions, etc., revealed that, in Grade 3, 65% was given to the use of integers, 4% to common fractions, and 17% to decimals. In Grade 6 the numbers were 36% integers, 0% fractions, and 54% decimals, the rest being devoted to other items like compound numbers. All this shows the predominant use of decimals in the problems of the pupils' experiences.

As to square and cube root, all computations relating to this line of work can now be done by tables or machines, and there is not the slightest practical use of teaching the subject to pupils in general, interesting though it may be for those who like mathematics.

If the future is to see eliminated much that is of doubtful value today, it is quite probable that the question of drill may be less serious—at least it will be less extensive in its applications. Professor Brownell gives some wholesome food for thought when he says: "The statement that the drill theory in its extreme form sets an impossible learning task for the child would seem to be justified." Probably the omission of unnecessary subjects for drill would be welcomed by those who would recognize the benefit of this means of fixing the remaining number facts and processes in mind.

It is not necessary to speak of the applied problems of arithmetic. They have changed from decade to decade as business customs and the needs of the home have changed. The best American arithmetics of today offer material that is both interesting and useful. If we are asked what can be substituted for the elimination of useless material already mentioned, the answer is an increase of practical and interesting problems.

One of the most encouraging contributions of the present is what is often called "the arithmetic of environment"—a rather pedantic term but a suggestive one. It means that arithmetic is unconsciously blending with economics and with a study of national resources. Thus we have problems classified as relating to our home industries and the great national industries as well.

This is possibly an outgrowth of the "project method" which was to revolutionize arithmetic but which often ran, as is so frequently the case with such efforts, to such an extreme as to be looked upon somewhat as a "mere fad." The use of real problems of the home, of its surroundings, of its near-by industries, and later, of the basic industries of our country naturally develops as geography does—from the near environment to the nation at large. Indeed, there is no reason why the arithmetic and the geographic curricula should not develop side by side in closer union than at present. In any case, the value of emphasizing the search for problems relating to children's interests will continue to be in evidence in the future as it is at the present time. The danger that this emphasis will lead children to be less interested in number while becoming more interested in its applications may be very apt to disappear with the gradual carrying of some of the simpler kinds of computation from a higher to a lower grade of the elementary school. In the upper grades, as already suggested, it is quite within the range of possibility that some inexpensive form of mechanical computation will gradually find its way from the store to the school, just as the typewriter has now a place in the high schools and as tables of interest, time, prices, powers, and roots, as well as commercial graphs, have a place in the upper grades of the elementary school, and as the "change-making" devices have found place in thousands of stores.

As to the high schools, even today many of them meet the demand for skill in stenography as well as in typing. There is also a demand today for skilled operators on computing machines of various kinds, and there seems to be no reason why the elementary schools should not have this in mind while preparing for the work in the high schools of the future. What Professor Brueckner says concerning illustrative material, including that of a commercial nature, shows a healthy tendency, and the ability to place some of this material in the hands of the pupil will follow in due time.

The objection that the disciplinary value of arithmetic is being abandoned because of the tendency to have the problems less puzzling has little foundation. Whatever discipline arithmetic has may better be secured from modern, informational, and interesting problems than from inherited puzzles. The latter should and probably will be treated as rewards rather than as punishments. There is no reason why we should not play with arithmetic just as we play with cross-word puzzles, dominoes, tennis balls, and the football. The recreations of mathematics are multitudinous, but they should be recreations, not tasks.

From our present point of view, therefore, the future seems to suggest to the high schools the development of mechanical computation, just as the present demands mechanical writing and bookkeeping. The amount of computation with pen and paper there, and even in the elementary grades, is bound to become less and less. The simple algebraic formula will find more welcome. The metric system is becoming commonplace in our kilocycles, meters, and kilometers. It would have made far more rapid progress if the schools had not tried to teach myriagrams, milliliters, and other terms which no one ever used. Even today Great Britain, a country which the New World looks upon as conservative, makes much more use of the simple metric units than we do. Perhaps we shall some day "catch up" with her.

The future will probably see more use of graphs of one kind and another than we see at present, graphs connecting up with decimal computations. The common fraction with denominators above 12, 16, and 32 will, as above suggested, give place to decimals, and the dull drudgery of working with fractions like $^{17}/_{21}$ will be forgotten.

The future will also recognize that the schools do not exist to make bookkeepers of all children, or to teach the arithmetic required by a course in mechanics or actuarial science. The mass of the children will learn what the mass of people need; the more elaborate mathematics will be for the specialist.

It is dangerous to prognosticate, but it is a pleasant pastime, and it is only by imagining a better future that the world makes progress.

THE MATHEMATICAL VIEWPOINT APPLIED TO THE TEACHING OF ELEMENTARY SCHOOL ARITHMETIC

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INTRODUCTION

By degrees the name "mathematics" is being applied to the school courses on the lower grade levels. Until quite recently "mathematics" referred only to high school subjects, but "junior high school mathematics" has now become almost universally the accepted name of the mathematics courses offered in Grades 7, 8, and 9. Work in the field of mathematics below the seventh grade, however, is still called "arithmetic." The fact that we do not classify arithmetic as elementary mathematics is not due to chance but rather to the way in which the subject of arithmetic has been treated.

It is the purpose of this chapter to consider elementary arithmetic from a mathematical point of view in contrast with what has been called a pedagogical point of view, and to offer evidence in favor of the mathematical viewpoint. What is meant by the mathematical point of view may be deduced from a statement by Bertrand Russell[1]:

> Mathematics is a study which, when we start from its most familiar portions, may be pursued in either of two opposite directions. The more familiar direction is constructive, towards gradually increasing complexity: from integers to fractions, real numbers, complex numbers; from addition and multiplication to differentiation and integration, and on to higher mathematics.

"Constructive, toward gradually increasing complexity" connotes a building up process based upon relationships. Guided by the mathematical point of view relationships assume supreme importance.

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[1] Russell, Bertrand, *Introduction to Mathematical Philosophy*, p. 1. The Macmillan Company, 1919.

At present the notion persists in certain quarters to the effect that the teacher and textbook writer necessarily must behave in certain unwise ways if the subject of arithmetic is treated as a branch of mathematics. In opposition to this notion is the contention that both the organization and the teaching of arithmetic can conform to sound pedagogical doctrine and at the same time follow the lines of mathematical development. The issue seems to be:

Should the aim in teaching arithmetic in the elementary grades be to develop ability to perform the mechanical procedures, after analyzing each procedure into its constituent elements of difficulty and then teaching every conceivable mechanical difficulty, and if time remains develop mathematical relationships?

or

Is it preferable to analyze the arithmetic of the elementary grades to determine the important and useful mathematical relationships, and then provide adequate teaching materials and instruction to develop an understanding of these relationships and, if there is sufficient time remaining after understandings have been developed and the simpler skills have therewith been taught, to teach the many other skills needed to perform all manner of mechanical operations?

Obviously those who would follow the second of the two programs take the mathematical point of view. Those who would adhere to the first program have in educational literature referred to it as the pedagogical viewpoint. In the literature dealing with the subject of arithmetic, certain objections to the mathematical viewpoint have been made. Chief among these is that a mathematical analysis of arithmetic produces an outline of learning elements which cannot be used effectively in the classroom. The learning elements thus derived are not, it has been charged, specific enough for learning purposes. There is also the assertion that the explanations and descriptions of processes resulting from mathematical analysis are on an adult level in the form of rules and definitions which are too brief and too compact and otherwise impractical for children to use. There seems also to be a feeling expressed by some that mathematical analysis does not permit the application of laws of learning because too much transfer between processes is demanded. All in all, the objections to the mathematical organization and presentation of materials of instruction seem to be based upon the notion that what is pedagogical and what is mathematical are not compatible.

Those who consider the learning elements of arithmetic as so many specifics which must be mastered without any relationship to one another deny that the human mind has the ability to classify, organize, arrange, and systematize. Whether we will it or not, children are constantly doing these things; they are sensing relationships and are utilizing them. Our methods of instruction which have neglected almost entirely the value of directed inferential thinking have driven children to generalize by confronting them with learning tasks and then relying upon the magic of repetition to bring about mastery. Those favoring the mathematical point of view point to the fact that, except in certain instances, children have mastered many of the difficulties of arithmetic by methods not known. As evidence, they point to the fact that many adults confess that they have retained habits formed during their early school days and that, consequently, at times they still make tens, refer to the addition doubles, and utilize other devices which are the products of inferential thinking. The extent to which children generalize in this fashion cannot be determined because when automatic response has been achieved the generalizations which were employed are forgotten. On this basis the claim is made that children generalize more than adults suspect. If children do these desirable things anyway, the proposal has been made that instructional material be so organized that generalizing shall come as a matter of course.

Furthermore, preference for the mathematical point of view has been based upon two other reasons: first, the average person has many more occasions to read and interpret quantitative data than he has to make actual computations, and, second, students who have been given a great deal of instruction in the mechanical procedures show, on the basis of test results, that they not only fail to retain the ability to perform the unrelated mechanical procedures correctly, but also fail to master them during the grade in which they are taught. The grade placement studies by Washburne[2] and others are cited as evidence for this statement.

Only a very few arithmetic studies have been reported which were designed to measure the value of learning by the inferential or generalized method. The writer has unpublished evidence and impressions obtained from the observation of pupils which, in addition to other studies which have been made, will be reported in this article.

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[2] Washburne, Carleton, "Mental Age and the Arithmetic Curriculum." *Journal of Educational Research*, pp. 210-232, March, 1931.

Part I

The Mathematical Viewpoint Applied to the Teaching of the Number Combinations

The foregoing discussion has been offered to indicate that opposite points of view exist regarding the extent to which opportunities for generalized learning should be provided in our schools. When an attempt is made to measure the effectiveness of this type of learning, two problems arise. One relates to the analysis of the content for teaching purposes and the other to the manner in which the analysis is to be employed in teaching situations. Obviously, the mathematician is primarily concerned with relationships. It is upon this basis that he would seek out the simple elements which form the foundation and follow with a building up process in which each new element grows out of simpler elements. In the teaching process he would be concerned with the manner in which the learning situations are presented and managed in the classroom after appropriate analysis has been made and the course of instruction has been organized according to the resulting analysis. It is not within the province of this chapter to make a complete mathematical analysis of the curriculum and to discuss the teaching thereof, nor does it seem necessary to defend the acceptance, for purpose of study, of those elements of arithmetic which find a place in our school courses. The first evidence to be submitted will deal entirely with the number combinations.

Addition combinations. Although the study to be reported involved the teaching of the one hundred addition and the one hundred subtraction combinations, only that which deals with the addition combinations will be reported in full. The experimental study was conducted in the Detroit schools over a period of two semesters, beginning in September and ending in June. Approximately three hundred beginning second grade pupils, the majority of whom had C, C −, D, and E intelligence ratings,[3] were selected for this study because the method of drilling upon specifics had yielded unsatisfactory results with this type of children.

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[3] These ratings are distributed on the normal curve of distribution according to the following percentages: A 8%, B 12%, C + 20%, C 40%, C − 20%, D 12%, E 8%.

From an inventory of generalizations made by children on their own initiative and from a preliminary experiment made during the year previous to the year during which this study was conducted, the one hundred addition combinations were placed in seven groups. They are presented in detail in Table I, to which reference will be made several times. The table not only indicates the grouping of the combinations for the generalized method experiment but also gives the per cents of error on the individual combinations for this experiment and for that conducted by Clapp[4] several years ago. In fairness to the Clapp experiment, it must be stated that in his tests every pupil tried every combination. In the Detroit experiment pupils were scored only on those combinations which they tried in the time allotted for the test. However, to obtain a fair measure on all the combinations, two test forms were used. Combinations in the second were placed in reverse order from the first test.

The addition groupings presented in Table I may be summarized as follows: (1) adding 1 and reverses, (2) adding 2 and the reverses, (3) adding 0 and reverses, (4) the doubles and one more and one less than the doubles, (5) the combinations of 10 and their reverses, (6) the combinations of 9 and their reverses, (7) miscellaneous combinations not included in the above and their reverses. Those familiar with arithmetic teaching will appreciate the difference between this grouping and that commonly found in textbooks and in arithmetic courses of study.

In the teaching of the addition combinations thus grouped, the teachers followed the general plan of introducing a given set of combinations through concrete experiences. The generalizations were demonstrated only if the pupils did not sense them. However, every effort was made to permit the pupils to do inferential thinking before any demonstrations were made. These demonstrations were not in the form of drills akin to phonetic drills in reading but were employed only in the initial stages of growth and thereafter referred to only in special cases. The demonstrations were for the purpose of assisting the slow-learning pupils in the acquisition of a method of combination attack when the organization of the materials of instruction did not suggest it to them. For example, if pupils did not, after knowing $2 + 2 = 4$, $3 + 3 = 6$, $4 + 4 = 8$, $5 + 5 = 10$, etc., sense that $2+3=5$, $3+4=7$, $4+5=9$, $5+6=11$, etc., the idea that the second set of combinations is either one more or one less than certain doubles was demonstrated. Thus the organization contained an intrinsic method of combination attack for pupils to grasp in the natural course of their number experiences. Mention may be made of the fact that the ability to sense relations was checked by having pupils describe the "trick" used or preferably by checking with larger numbers. The "one more or one less than a double" idea was checked by such problems as $20+20=40$, $20+21=?$

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[4] *The Number Combinations, Their Relative Difficulty and Frequency of Their Appearance in Textbooks*, p. 20. Bureau of Education Research Bulletin No. 1. University of Wisconsin, 1924.

It is one thing to experience new combinations and to generalize about them and another to master them to the point of automatic response. The fact that many children were able to write sums at the rate of seventeen per minute on the final test is partial evidence that automatic response was achieved. In keeping with the plan of stressing generalizations rather than specific combinations, strengthening and remedial exercises were organized. Following each drill test the individual pupils located in turn the group to which each incorrect combination belonged. The group had been listed on the blackboard in the form of a chart as each new group was introduced. Thus practice was given in making recognitions which served as cues to action. As a corrective measure, drill was on the "kind of combination" missed rather than on the specific combinations. In other words, pupils did not, after missing a combination, "write the correct form ten times," study it aloud, play flash-card games, or engage in repetitions of the specifics of learning, but they did refresh their minds on methods of attack.

At the close of the second semester the pupils were given a written test form of the one hundred addition combinations arranged in mixed order. As has already been mentioned, 6 minutes were allowed for the writing of the one hundred sums. No initial test was administered because some pupils could not read numbers, and only a few could write numbers upon entering the second grade. Furthermore, results were at hand for purposes of comparison from another experiment made in the Detroit schools in which second grade children had religiously followed the method of drilling on specifics. In that experiment pupils used workbooks in which spaces were provided for recording and writing the correct forms of combinations missed in tests. Also, in the workbook experiment the Clapp order of difficulty was employed in the grouping of the combinations.

TABLE I

THE GROUPING OF THE 100 ADDITION COMBINATIONS FOR THE GENERALIZED METHOD EXPERIMENT WITH THE DETROIT AND THE CLAPP[5] PERCENTAGES OF ERROR FOR EACH COMBINATION

1. Adding 1

	1	2	3	4	5	6	7	8	9
	1	1	1	1	1	1	1	1	1
Detroit %	1	11	6	9	4	5	4	5	3
Clapp %	5.8	4.5	7.9	7.7	13.1	7.1	9.7	10.8	10.2

The reverses of adding 1

	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9
Detroit %	1	8	4	8	8	7	6	8	7
Clapp %	5.8	11.5	5.6	10.1	9.3	6.4	7.1	7.1	10

2. Adding 2

	2	3	4	5	6	7	8	9
	2	2	2	2	2	2	2	2
Detroit %	3	8	11	9	8	2	3	6
Clapp %	3.8	8.5	11.4	14.9	12.2	10.8	10.8	7.2

The reverses of adding 2

	2	2	2	2	2	2	2
	3	4	5	6	7	8	9
Detroit %	6	11	10	10	6	6	6
Clapp %	14.1	10.1	10.5	19.3	15.8	8.1	9.8

3. Adding 0

	1	2	3	4	5	6	7	8	9
	0	0	0	0	0	0	0	0	0
Detroit %	6	7	8	6	8	4	7	7	10
Clapp %	12.3	7.7	12.5	7.3	8.8	9.9	13	12.7	14.1

The reverses of adding 0

	0	0	0	0	0	0	0	0	0
	1	2	3	4	5	6	7	8	9
Detroit %	19	17	19	18	13	15	18	18	26
Clapp %	10.9	11.5	12.4	12.6	11.9	14.2	13.1	9.8	12.8

4. The doubles

	2	3	4	5	6	7	8	9
	2	3	4	5	6	7	8	9
Detroit %	3	8	2	2	1	6	3	2
Clapp %	3.8	4.4	8.3	3.0	9.7	8.3	9.6	8.2

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[5] *Ibid.*

TABLE I—(*Continued*)

One more than the doubles

	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	–	–	–	–	–	–	–
Detroit %	6	9	6	9	9	14	7
Clapp %	14.1	14.3	11.1	19.8	24.5	30.5	30.2

One less than the doubles

	3	4	5	6	7	8	9
	2	3	4	5	6	7	8
	–	–	–	–	–	–	–
Detroit %	8	7	9	10	8	11	9
Clapp %	8.5	11.1	8.2	16.3	20.8	34.9	24.2

5. Adding to 10 and reverses *

10	10	10	10	10	10	10	10	10
1	2	3	4	5	6	7	8	9
—	—	—	—	—	—	—	—	—

(Not included in Clapp list or in Detroit combination test.)

1	2	3	4	5	6	7	8	9
10	10	10	10	10	10	10	10	10
—	—	—	—	—	—	—	—	—

(Not included in Clapp list or in Detroit combination test.)

6. Adding to 9 and reverses

	9	9	9	9	9	9	9	9
	1	2	3	4	5	6	7	8
	–	–	–	–	–	–	–	–
Detroit %	3	6	12	12	11	8	9	9
Clapp %	10.2	7.2	14.7	22.9	22.5	26.0	36.7	24.2
	1	2	3	4	5	6	7	8
	9	9	9	9	9	9	9	9
	–	–	–	–	–	–	–	–
Detroit %	7	6	10	11	12	9	12	7
Clapp %	10.0	9.8	12.0	21.8	25.1	32.5	39.8	30.2

7. Miscellaneous combinations and reverses

	5	6	6	7	7	7	8	8	8	8
	3	3	4	3	4	5	3	4	5	6
	–	–	–	–	–	–	–	–	–	–
Detroit %	7	8	9	4	7	10	3	8	12	12
Clapp %	15.1	16.5	12.3	16.2	15.8	23.3	12.3	17.3	33.3	32.5
	3	3	4	3	4	5	3	4	5	6
	5	6	6	7	7	7	8	8	8	8
	–	–	–	–	–	–	–	–	–	–
Detroit %	8	7	8	8	8	11	9	11	10	12
Clapp %	14.9	13.0	21.4	18.9	20.0	24.2	14.4	16.2	30.9	33.8

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* Ordinarily not included in the one hundred addition combinations. The combinations of 9 were related to those of 10 in this experiment and therefore taught.

Thus there were differences between the two experiments in the grouping of the combinations and in the teaching methods. The comparative scores on a six-minute test of the one hundred addition combinations together with other significant data are herewith offered. The generalized-method pupils were taught by four teachers, three of whom taught two classes each. The pupils in Group I in the Specific Drill report were taught by seven teachers. The Group II pupils were selected from six classes racially and otherwise comparable with Group II of the generalized-method experiment. The Group I children in each experiment were comparable mentally and in other respects. Comparative results of the two experiments follow in Table II.

TABLE II

AVERAGE PER CENT CORRECT OF THE 100 ADDITION COMBINATIONS BY GROUPS, WITH THE INTELLIGENCE DISTRIBUTION OF EACH GROUP

Experiment	Average Per Cent Correct	No. of Pupils	Intelligence Distribution, in Per Cents						
			A	B	C+	C	C−	D	E
Generalized Method									
Group I	99	45	17	33	31	19			
Group II	76	217		6	7	13	30	29	15
Specific Drill									
Group I	77	227	13	19	18	27	14	5	3
Group II	55	66	4	8	17	20	18	15	17

The final test scores speak for themselves. It must be recalled that both methods were tried under experimental conditions and that the intelligence ratings were for the most part decidedly in favor of the specific-drill groups. Despite this fact, the generalized-method group results were superior to those obtained by methods of specific drill. Comparable group comparisons are decidedly in favor of the generalized method.

The effectiveness of teaching the addition combinations grouped according to generalizations may be realized from a further study of the percentages of error recorded in Table I.[6] Inspection of the percentages of error within the groupings used for teaching purposes seems to give support to the contention that combinations can be grouped effectively according to generalizations. Special attention is also called to the following facts:

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[6] See Table I, page 218, for per cents of error on each of the one hundred addition combinations taught by the generalized method, and the Clapp per cents of error.

a) The reverses of combinations were on the average about equal in difficulty to the basic combinations.

b) The limits of error within groups fall within a narrow range.

c) The doubles, adding 0, adding 1, and adding 2, were somewhat easier than the other combinations.

d) The percentages of error for the so-called "hard" combinations were not much greater than the percentages of error for the so-to-speak "easy" combinations.

The last point is brought out very forcibly when Clapp's figures for the difficulties experienced in learning the combinations are organized according to the groupings employed in the generalized-method experiment.[7]

Although the generalized-method and the Clapp experiments were not managed in the same way (and consequently the scores cannot be compared directly), there is substantial agreement between the two experiments in the relative difficulty percentages of error for the doubles group, adding 1, adding 2, adding 0, and their reverses. There are, however, marked differences between results obtained for the so-called "harder" combinations. The pupils who employed the generalized method learned the combinations of larger numbers almost as well as those of the smaller numbers. For example, $7 + 9$ and $9 + 7$, which are listed as the most difficult of all the addition combinations on Clapp's list, were not any more difficult than $5 + 2$, $4 + 2$, and $9 + 3$ for the pupils who were studying according to the generalized method.

The disparity between the order of difficulty of the one hundred addition combinations obtained by Clapp and that resulting from the generalized method of organization and instruction is marked. A coefficient of correlation of $+ .52$ between the orders of difficulty indicates quite clearly that the difficulty experienced in mastering the addition combinations is not a fixed matter. Likewise it seems to substantiate the claim that the extent to which relationships among the combinations are built up does influence the mastery of them.

Attention is called to the fact that the studies by Washburne and Vogel,[8] Counts,[9] Smith,[10] and Phelps,[11] the results of which tallied

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[7] See Table I.

[8] Washburne, C. W. and Vogel, Mabel. "Are Any Number Combinations Inherently Difficult? *Journal of Educational Research*, 17:235-255. April, 1928.

[9] Counts, George S. *Arithmetic Tests and Studies in the Psychology of Arithmetic.* Supplementary Educational Monographs, No. 4. University of Chicago Press, 1917. [*Footnotes continued on page 222.*]

with Clapp's order of difficulty of the combinations, were not planned to test the effectiveness of an organization and a type of instruction aiming to emphasize number relationships.

Even in Olander's [12] study, which purported to measure transfer of learning, the pupils were not led to recognize intimate relationships until the end of the fifth week. Furthermore, the groupings employed in Olander's study followed the Clapp order of difficulty. Also, the recognition of generalizations was incidental rather than basic during the introductory and mastery periods of learning. The extent to which transfer plays a part obviously depends upon the steps taken to obtain transfer. In Olander's study much was made of generalizing but possibly much of the value therefrom was negated from drill on the specifics of learning.

Summary. From the foregoing it seems reasonable to conclude that training in the art of generalizing through the employment of a method which is entirely directed toward and based upon the pupils' consciousness of number relationships does change the whole complexion of the situation as it relates to a mastery of the number combinations. The data obtained under comparable conditions as herein reported seem to indicate a superiority of the generalized method over a repetitive drill method for superior as well as for duller pupils. Also, combinations which have been presumed to be difficult prove to be learned as well as those which have been thought to be easy. Furthermore, the evidence at hand seems to indicate the possibility of extending the mathematical point of view to the whole field of arithmetical learning.

The advisability of applying the mathematical viewpoint to the organization and teaching of the subtraction, multiplication, and division combinations has been subjected to some experimentation. Data from which reliable conclusions can be drawn are not as yet available. Observations of teaching procedure aiming to cause children to sense and to build up number relationships have, however, yielded what might be termed tentative conclusions favoring that method.

Subtraction combinations. The teaching of the subtraction

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[10] Smith, James Henry. "Arithmetic Combinations." *Elementary School Journal*, 21:766-770, June, 1921.

[11] Phelps, C. L. "A Study of Errors in Tests of Adding Ability." *Elementary School Teacher*, 14:29-36, September, 1913.

[12] Olander, Herbert T. "Transfer of Learning in Simple Addition and Subtraction." *Elementary School Journal*, 31:358-370, 427-437, January-February, 1931.

combinations was coupled with the teaching of the addition combinations in the study already reported. The data from the subtraction tests are not as complete as those obtained from the addition tests but are sufficiently complete to be offered for consideration. In so far as it was possible in the subtraction experiment, the addition generalizations were employed. The addition combinations of 10 seemed to be favored by the pupils in obtaining answers for many of the subtraction combinations. For example, to $17 - 9$, many pupils responded: $9 + 1 = 10$, $10 + 7 = 17$, $9 + 8 = 17$, $17 - 9 = 8$; and to $14 - 8$ the steps were: $8 + 2 = 10$, $10 + 4 = 14$, $8 + 6 = 14$, $14 - 8 = 6$.

The subtraction results from the generalized-method and specific-drill experiments, for which the addition scores were presented in Table II, follows in Table III.

TABLE III

AVERAGE PER CENT CORRECT OF THE 100 SUBTRACTION COMBINATIONS BY GROUPS WITH THE INTELLIGENCE DISTRIBUTION OF EACH GROUP–TIME 6 MINUTES

Experiment	Average Per Cent Correct	No. of Pupils	Intelligence Distribution, in Per Cents						
			A	B	C+	C	C−	D	E
Generalized Method									
Group I	91	45	17	33	31	19			
Group II	65	217		6	7	13	30	29	15
Specific Drill									
Group I	54	227	13	19	18	27	14	5	3
Group II	38	66	4	8	17	20	18	15	17

Although the statistical reliability of the differences between the average scores of the generalized-method and specific-drill groups on the six-minute subtraction test are not available from the data at hand, the generalized-method scores seem by comparison to be even better for subtraction than for addition.

Multiplication and division. The plan for teaching the multiplication combinations by the generalized method can only be indicated at this time. An experimental effort has been made to employ the basic principles of organizing the multiplication combinations according to generalizations and to conduct the learning experiences from the point of view of utilizing relationships. For example, the pupils have been stimulated to build up their own multiplication tables by adding; the completed tables have been studied for the purposes of noting characteristic relationships and establishing points of departure.

The table of 9's seemed to present the largest number of opportunities for generalizing. Pupils soon discovered that the right-hand digits of the products descended in regular order from 9 to 0 and the left-hand digits beginning with the 1 of 18 ascended to 9 of 90. A few pupils noted the fact that the sum of the digits in each product is 9. As a reference point 45 was chosen by many pupils because they were aware of the fact that it was the product of 9 × 5. From this they readily obtained the products of 9 × 6, 9 × 7, 9 × 8, and 9 × 9. A few sensed 9 × 9 to be 81 because 9 × 10 = 90. Many pupils voted the table of 9's to be next in difficulty to the 2's and 5's. The introduction of the table of 9's following the 5's seemed to promote the habit of searching for generalizations, which habit was applied to the study of the other tables.

Obviously the pupils following the generalized method spent a greater amount of time than usual in class discussion and activities aimed to develop a relationship among the multiplication combinations. Throughout, the reverse forms were identified as new tables were introduced. Also, the time spent by the teachers on drill exercises in the form of games, flash-card drills, and other forms of repetition was much less than is ordinarily devoted. While data are not at hand from which any conclusion can be drawn, the pupils did develop methods of attack which eliminated much of the guessing so common when children are in doubt. Just as subtraction was closely linked with addition, division was related to multiplication.

Attention is again called to the basic idea of providing experiences which will cause pupils to sense number relationship and thereby enable them to develop a mode of attack rather than leave this to chance. Teaching by this method shifts the emphasis from excessive drilling to study based upon understanding.

PART II

THE MATHEMATICAL VIEWPOINT APPLIED TO ARITHMETIC TOPICS SELECTED FROM THE MIDDLE AND UPPER GRADES

The possibilities of applying the mathematical point of view to the teaching of arithmetic beyond the fundamental number combinations are apparent. However, the fact that all arithmetic represents a system of interrelationships makes it quite impossible to enumerate all of them. The more one studies the science of arithmetic the larger the number of relationships becomes. There are, however, a few stumbling-blocks in the path of pupils from the lower grades to the higher which deserve special consideration. It is with the idea of suggesting an extension of the mathematical point of view to include these topics that some of them will be discussed.

Fractions and the operations with fractions. Traditionally the study of the common fractions has dampened the ardor of hosts of boys and girls for the subject of arithmetic in our American schools. This fact is not surprising when many of our standard textbooks and much of the present-day teaching emphasizes terms such as numerator and denominator, proper and improper fractions, common divisors, and common denominators more than the understanding of the basic idea of fractions. Teachers are not to be censored too severely for this if the space devoted in the textbook to the development of the fraction concept is cut to a minimum so as to provide room in the textbook for the many specific difficulties into which the topic has been analyzed. Furthermore, the absence from the classroom of concrete materials and tools in the form of scissors, drawing instruments and containers, colored paper, string, strips of cloth and wood, etc., makes the task of developing the fraction concept a difficult one. It is not surprising, then, that many children actually believe $1/16$ to be more than $1/2$ as a result of the type of instruction they receive. The reason for this misconception is obvious. Also, many children who can manipulate fractions to the extent of subtracting with borrowing do not sense an equality between $3/4$ and $6/8$ nor can they distinguish $3/4$ as 3 of the 4 equal parts of a whole. Contact with and observation of children in the fifth and sixth grades will bear out these contentions. These faults are clearly the result of not approaching the teaching of fractions from a mathematical point of view. A consideration of the teaching of the four operations with common fractions will serve further to illustrate this point.

An informal experiment has been conducted by the writer to determine the advisability of first teaching addition, subtraction, multiplication, and division of common fractions without introducing such devices as those for finding common divisors, cancelling, and inverting divisors. Naturally, before these devices were introduced only those simple fractions which could be managed mentally or with objects were involved. The so-called devices for changing to higher or lower terms, reducing improper fractions, finding common denominators, cancelling and inverting divisors were presented only when the work could not be performed mentally. The teachers who were engaged in this experiment were unanimous in their opinion that the pupils in the experimental groups worked with more confidence and less confusion of thought than pupils ordinarily do.

The multiplication and division process ideas, as one might well assume, proved to be the most difficult to manage. In order to bridge the gap between whole number relationships and fractional relationships in multiplication situations, the "of" idea was eliminated from all consideration. Instead, exercises such as the following were employed:

(*a*)	(*b*)
$6 \times 4 =$	4 times 6 =
$6 \times 3 =$	3 times 6 =
$6 \times 2 =$	2 times 6 =
$6 \times 1 =$	1 times 6 =
$6 \times \frac{1}{2} =$	½ times 6 =

(*c*)	(*d*)
$\frac{1}{2} \times 1$	$1 \times \frac{1}{2}$
$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{2}$
$\frac{1}{2} \times \frac{1}{4}$	$\frac{1}{4} \times \frac{1}{2}$
$\frac{1}{2} \times \frac{1}{8}$	$\frac{1}{8} \times \frac{1}{2}$

In the course of instruction under discussion, " ½ times as much" took the place of " ½ of." To find " ½ of" a number it was divided by 2. Thus an attempt was made to establish a multiplication relationship between fractions and between fractions and whole numbers.

The division idea was extended from whole-number situations to fraction situations in much the same way. In the first place, instead of saying, "4 divided by ½" for $4 \div \frac{1}{2} =$, pupils used the whole number vocabulary in the form of, "How many ½'s in 4." In a similar manner pupils determined "How many ¼'s in ½ " and "How many ½'s in ¼," etc. Exercises of the following type were employed to link the division idea with fractions and thus lead to the generalization that the result of a division of any number by a fraction is greater than the number operated upon. Exercises of the following types were employed to extend the notion of division relationships from whole number to fraction relationships.

(*a*)	(*b*)	(*c*)	(*d*)
$24 \div 4 =$	$\frac{1}{2} \div 2 =$	$\frac{1}{2} \div \frac{1}{2} =$	$\frac{1}{2} \div \frac{1}{4} =$
$24 \div 3 =$			
$24 \div 2 =$	$\frac{1}{4} \div 2 =$	$\frac{1}{4} \div \frac{1}{2} =$	$\frac{1}{4} \div \frac{1}{4} =$
$24 \div 1 =$			
$24 \div \frac{1}{2} =$	$\frac{1}{8} \div 2 =$	$\frac{1}{8} \div \frac{1}{2} =$	$\frac{1}{8} \div \frac{1}{4} =$

Obviously the exercises designed to stress division relationships between common fractions and between common fractions and whole numbers included only very simple fractions. The method of inverting the divisor and multiplying was first applied to problems which could also be solved mentally and thus confidence in the new method was developed.

Mixed numbers. Observation of fifth and sixth grade pupils seems also to suggest that the whole numbers and the fractions of mixed numbers have not been brought together in a meaningful way. As an illustration, many pupils have not developed the trick of approximation and consequently are satisfied with such answers as $490\frac{1}{2}$ and $48\frac{1}{2}$ for the problem $6\frac{3}{4} \times 8\frac{2}{3} = ?$. This lack of judgment is even more evident in the case of multiplication of mixed decimals. For a problem such as $7.5 \times 4.6 = ?$, 3450 and 345 are not infrequent answers. These illustrations seem to indicate that the values of mixed numbers are not associated with the number scale. Very little instruction was required to cause children to appreciate that $6\frac{3}{4}$ is almost 7 and $8\frac{2}{3}$ is almost 9, and therefore the product of these two mixed numbers would be between 48 and 63. In the case of 7.5×4.6, answers like 3450 and 345 indicate quite clearly that 7.5 and 4.6 were not sensed as more than 7 and less than 8 and more than 4 and less than 5, respectively. Only a small amount of instruction based upon the foot rule or the number scale was required to remove this difficulty. These illustrations are offered to stress the fact that provision must be made to build up the number relationships so important to students of arithmetic.

Comparison of numbers. The importance of this topic is not appreciated by many arithmetic teachers and textbook writers. Evidences of an almost total lack of ability to compare or to relate whole numbers one to another may be ascertained by questioning the average seventh or eighth grade class. When asked what part of 4 is 8 or how many 8's are there in 4, most children will reply that there are no 8's in 4 because 8 is larger than 4. Thus the idea of relationship between whole numbers breaks down when the resulting ratio is less than 1.

The absence of the ratio concept is also apparent when junior high school children are asked to find what per cent one number is of another when the per cent is 100 or more. The majority of eighth grade pupils tested informally gave either 50% or 33⅓% as an answer to the problem: "A pair of fancy pigeons was bought for $4 and sold for $12. What was the per cent gain of the selling price over the cost?"

Asked how they would find the tax rate when the total budget and assessed valuation figures are known, many teachers have admitted that they would divide the smaller number by the larger one. Such deductions are natural when rules such as the following may be found in mathematics textbooks: "To find what fractional part one number is of a larger one, write the smaller number over the larger one to make a fraction. Then reduce the fraction to lowest terms."

The effect of teaching of this type, which seemingly is concerned more with immediate success in computation than with basic understanding, is obvious.

The whole and its parts. From the second grade through the eighth grade evidence may be obtained which indicates that the relationship between the whole and its parts is not sensed. For example, many third grade children find it difficult to obtain the number of words correct in a spelling test if the total number of words and the number wrong are known. Similarly, many fifth grade children are confused when given, for example, the part of the pie remaining and asked to find the part eaten. In a like manner teachers of many slow sections are forced to omit finding the net price by multiplying by the per cent left after a discount has been taken. The difficulties arising in the teaching of upper grade work seem to indicate, for example, that the extension of the formula $a + b = c$ to $a = c - b$ and $b = c - a$ does not come by chance but must, like the many other important number relationships, be made goals toward which arithmetic instruction should be pointed.

CONCLUSION

To the list of illustrations offered to show the ill effects resulting from teaching arithmetic without the thought of building up number relationships, more could be added. The prime purpose of the writer, however, has not been to present a complete program of arithmetic organized according to the mathematical viewpoint nor to indicate how the learning situations might in all cases be managed, but rather to present a point of view. The beginnings of experimentation in the lower grades have been reported as well as impressions received from dealings with children in the middle and upper grades. The writer does not claim that any conclusive deductions can be drawn. However, it seems that when the value of mathematics teaching in the secondary schools, by the memory and specific-drill methods, is being questioned, it would be well to forestall similar attacks on the mathematics of the lower schools.

Already there are those in the teaching profession who would spend less time teaching arithmetic because studies have been made showing how little arithmetic is used by housewives and others. These reductionists have not, however, proposed to determine what arithmetic people might use to advantage in situations in which they find themselves. In other words, the possibility of improving the responses made to the situations of everyday life has not been considered by the reductionists.

Those holding the idea that arithmetic is a set of specifics upon which boys and girls must be drilled are unwittingly lending support to the reductionists by listing the exact number of skills which must be mastered. It would, indeed, not be difficult to prove that a large percentage of the specific skills and facts listed are not used by the average citizen and on this basis recommend further reduction. The results of the mental age study by Washburne[13] strengthened the idea held by many that in the courses of instruction in elementary school arithmetic there is still much dead wood which should be removed. Some of the skills, such as the more difficult cases of long division, much of the multiplication and division of common and decimal fractions, and a large part of the work with denominate numbers, might well be eliminated in the interest of more understanding. However, the part that arithmetic study might play in the total education of the child is such that the thought and understanding phases of arithmetic should be given more rather than less time in the curriculum.

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[13] See footnote 2, page 214.

In contrast to the idea that arithmetic is nothing more than a list of specific skills is the one to which support has been given in this chapter, namely, that arithmetic is more than a set of specific skills and facts—it is a mode of thought resulting from an appreciation of and an awareness of the many interrelated elements within the system of numbers. The individual who has either from school instruction or from his own reflection brought together the elements of arithmetic into a system of number relationships is not lacking in quantitative sense. In the utilization of the tools of arithmetic he does not depend upon the memory of a rule or a fixed pattern but senses the quantitative aspects of a situation and proceeds to manipulate numbers much as a linguist gives forth words. If a convenient method of satisfying the requirements of a situation does not present itself, he is apt to invent one. Although a certain percentage of the school graduates do acquire a quantitative sense, they are seemingly doing it in spite of the school which directly does very little to develop it.

Almost forty years ago McLellan and Dewey gave voice to the idea that in teaching arithmetic relationships should be stressed. The idea was expressed as follows:

> An education which neglects the formal relationships constituting the framework of the subject matter taught is inert and supine. The pedagogical problem is not solved by railing at 'form' but in discovering what kind of form we are dealing with, how it is related to its own content, and in working out the educational methods which answer to this relationship.[14]

If this type of arithmetic education were put into effect, several adjustments would have to be made in the majority of our American schools.

First, provision would have to be made for pupils to discover relationships. This obviously is contingent upon a grasp of the fundamental quantities with which pupils will deal and of the process ideas by means of which these quantities are related. In many of our American schools much more present-to-sense experience than is now provided would be required. However, there are educational groups which carry this practice to the point of diminishing returns. The idea of a possible concrete basis should carry over from quantities and processes which have been studied in the concrete to extensions and modifications of these quantities and processes met in the abstract. It seems sufficient to carry this matter to a point that the pupil will know that a concrete content may be given to the quantitative abstraction he has acquired if he chooses to do so.

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[14] McLellan, J. A. and Dewey, John, *The Psychology of Number*, p. xii. D. Appleton and Co., 1895.

Second, if the new is to be apprehended in terms of the old, the arithmetic of the elementary grades will have to be simplified. Many teachers rightly contend that the arithmetic courses for elementary school children (Grades 1 to 6) include so many skills that there is little time to teach for understanding. Now that children remain in school, generally speaking, through the eighth or ninth year, there is little need for crowding the whole arithmetic curriculum, including the topics of percentage and interest, into the first six years of schooling. Perhaps the grade placement studies now under way will influence textbook writers and course-of-study makers to simplify the arithmetic curriculum for the elementary grades. Perhaps the major portion of time set aside for teaching arithmetic in the elementary grades can be spent to advantage on the thought and understanding aspects of the subject.

Third, should a shift in emphasis be made from teaching skills, as such, to developing understanding, the present methods and facilities for teaching would require considerable revision. Pupils necessarily would work more with concrete materials than they now do. Teaching materials for the arithmetic classroom might then cost much more than they do at present. The traditional classroom organization and arrangement would need to be modified. In all probability teaching arithmetic from the mathematical point of view would require much more skill and training on the part of teachers than is now demanded. It is also conceivable that pupils would derive much more satisfaction from their learning experiences than they now do. Finally it seems to be self-evident that with a background of understanding the skills would, when introduced, be mastered with much less effort and with more profit. Also, the social applications and settings would play a larger part than they do at present if understanding were sought.

The possibility of completely revamping the present elementary school arithmetic program of teaching after the fashion which has been described is quite remote. Our educational traditions as they relate to school administration and classroom practices are too strong to permit much variation from present practices. There is, however, the possibility even now of basing the teaching of the skills upon a foundation of understanding even though many of the skills are, as many think, taught prematurely.

There are indications that suggest that further simplification of the arithmetic content of the lower grades is now taking place. The introduction of certain topics of arithmetic, such as long division and the operations with common and decimal fractions, is being delayed in some school systems. These changes have resulted mainly because the testing movement has indicated a low degree of mastery. The basic reason for low mastery may, however, be due to an absence of basic understanding of the number relationships involved.

The probable course of events in the elementary schools can be predicted by referring to that which is taking place in the secondary schools. A process of simplification and modification of the elementary algebra curriculum has been in operation for more than a decade and it has been in the direction of substituting understanding for the complicated skills of algebra. Those familiar with happenings in the field of elementary algebra and who are in sympathy with the new movement see in it the strongest argument for teaching algebra. Likewise those interested in the teaching of arithmetic may well realize that arithmetic will be attacked just as algebra and geometry have been unless the real values of the subject are sought.

The evidence at hand seems to give support to the contention that it is only by taking the mathematical point of view with respect to the teaching of arithmetic that the potential values of the subject can be derived from a study of it. On that basis can the teaching of arithmetic be justified and demanded.

THE NEW PSYCHOLOGY OF LEARNING

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The significance of the New Psychology for education cannot be sufficiently understood without a brief historical sketch. Not only psychology, but all science, including mathematics, is subordinate to a Universal Cultural Pattern or World View. More than that, this world view has fluctuated cyclically, between two extremes, one in which interest has been in the part-whole problem, and the other, in which interest has centered almost exclusively on parts. In the one case such problems came to the front as the rôle of the part in the whole. In biology, for example, the uppermost question was: What is the function of the part in contributing to the economy of the organism as a whole? In chemistry, the problems of the atom versus the molecule, substance-chemistry versus stereochemistry, rose into the foreground. In mathematics, during such periods, interest centered on geometry, series, transformations, rotations, groups, form, invariance, and maxima and minima, that is, on problems of mathematical wholes or systems. In psychology, main problems were presented in terms of patterns, selves, personalities, and social adjustment. In social science, the main problem was the relation of the individual to the group. Society was defined as something more than the sum of its individuals. The emphasis was socialistic. Coöperation was regarded as a basic process in nature and all science became teleological. Philosophy was rationalistic and idealistic. Three periods such as these have stood out with extraordinary clearness, centering around the dates 1250, 1650, and 1820, more especially the latter two.

History of the New Psychology. Following each of these periods, for reasons that cannot here be discussed, the general culture pattern swerved toward an exclusive interest in parts, elements, and individuals. These were periods of mechanistic science, emphasis upon competition as a basic force in nature, individualism, revolution, expansion, imperialism, warfare, exploration. Wholes were then mechanistically regarded as mere aggregates of parts. Mathematics turned away from geometry, particularly from 1740 to 1800, and centered upon analysis, infinitesimals, and probabilities. Ethics became utilitarian and hedonistic. Philosophy turned to materialism, voluntarism, and romanticism. These were periods of laissez-faire. Individuals were emancipating themselves from tyranny, or were grasping for power. The dates 1400, 1775, and 1860 represent the peaks of these mechanistic movements. Notice that in the history of human endeavor, these are the periods when thought has concentrated upon the part, the element, the individual, as if a part did not belong to a whole, at all. The names of such persons as Paracelsus, Vives, Machiavelli, Rousseau, Hume, de la Mettrie, Diderot, Adam Smith, Jeremy Bentham, J. S. Mill, Darwin, Virchoff, Haeckle, Büchner, Malthus, Lagrange, Weismann, Dalton, Boltzman, Prout, Schleiden, Schwann, and Nietzche, typify mechanistic thought, while those of Thomas Aquinas, Leibnitz, Harvey, Casper Wolff, Grotius, Campenella, St. Simon, Proudhon, Lilienfeld, Worms, Robert Owen, Cuvier, von Baer, Claude Bernard, Berthollet, Avogadro, Gerhardt, Butlerow, Wilhelm Ostwald, Clerk Maxwell, Ernst Mach, Chasles, Steiner, Lobachevsky, Galois, Riemann, Dedekind, Hilbert, Kant, Hegel, Bradley, Alexander, and Bosanquet, all typify the opposite approach.

A study of history shows that times in which constructive thought was achieved, relative to the part-whole problem, were for the most part confined to the first three periods mentioned, represented by the years 1250, 1650, and 1820. These were vitalistic and idealistic periods. The years 1400, 1775, and 1860 came in the midst of materialistic and atomistic periods. During these latter periods theories of nature developed which were for the most part barren, because overinterest in the part, the element, and the infinitesimal blinded science to the ever-present problem of form, wholes, configuration, unity, system, and order, in short, to the problem of *the part in the whole.* So, in these periods, the true nature of the part-whole problem was neglected. The world was regarded as a fortuitous concourse of atoms. Teleology and over-summative wholes were denied. These were periods of applied science and of material, rather than spiritual, progress.

It was during the atomistic, mechanistic, materialistic, romantic, and expansive mid-Victorian era that our present educational system was conceived and executed, and it was under the influence of this cycle that our "orthodox" educational psychology, the psychology of association, bonds, conditioned reflex, and behaviorism, was actually born. So our schools are constructed as if the main problem of life was one of parts. We have a departmental system in which one subject is studied out of relation to the other subjects, as an *isolated* discipline. Such a method is artificial. To make a long story short, *our present educational system, methods of teaching, and supposed laws of learning are not based upon the main trend of scientific thought down through the centuries.*

If one goes back to the roots of current scientific theories found in relativistic physics, organismic biology, and Gestalt psychology, he will discover them almost exclusively in the periods, 1630 to 1740, 1800 to 1840, and in the thirteenth century, hence back to Aristotle and Plato. Generally speaking, the scientific conceptions prevailing from 1740 to 1800 and from 1840 to 1900 (mathematics excepted) constituted unproductive sidetracks. They were prejudiced conceptions, coming during emotional rather than intellectual eras. Unfortunately, the atomistic, superficial, distorted conceptions of nature that held sway during the latter period shaped our educational system, objectives, and methods, and still dominate our textbooks. Repetition, drill, departments, grades, and the like, reflect the atomistic, mechanistic, and hedonistic cul-de-sac in which human thought has lost itself twice, in the later 1700's and again in the middle 1800's. Errors become institutionalized and acquire inertia. Today we face the huge problem of ridding our school system of these errors and of the theory behind them.

This, then, is what the modern attack on the school system is all about. We are returning to the main cultural track again. Had we been students in the days of Steiner, von Staudt, Lobachevsky, Cuvier, von Baer, St. Simon, Herbart, Kant, and Hegel, around one hundred years ago, we would understand the modern movement much better. As it is, in order to pick up the continuity of the main track, one must skip over the assumptions, *not the data*, of Watson, Thorndike, and Titchener, and go back to Herbart and Hamilton; over Weismann, Haeckel, and Darwin, to von Baer and Cuvier; and over "rugged individualism" to the more social views of the opening of the century. Even Karl Marx belongs to Darwin, not to the truly social point of view—the correct view of the laws of social wholes. One must skip over J. S. Mill, Bentham, Ricardo, and Malthus and go back to Althusius, Campenella, and Grotius.

There is no need of groping with our current problems. A clear conception of methods and objectives is quite possible. There is nothing difficult about them, except the labor of eliminating the inertia of mid-Victorian scientific mores and taboos.

Today, educators have a distorted conception of how to arrive at facts. They seem to think that all one needs to do is to set up an experiment, to devise a test, or to send out a questionnaire. They are thinking in terms of 1860, or else they are not thinking at all.

It is hard to believe that our immediate heritage, as regards basic scientific conceptions, is mostly wrong. But what else would have plunged us into so violent a revolution as that of giving up in physics a mechanistic and absolute conception of the universe for an organic and relativistic one; or in biology of giving up the mechanistic cell theory for an organismic one; or in psychology, association for Gestalt; in social problems, rugged individualism for a collectivistic point of view; and in ethics, utilitarianism for idealism? The facts are that the whole structure of mid-nineteenth century thought was wrong, because it put the part first, and that this thought shaped the character of the psychology we learned, the school system we attended, and the method of instruction to which we were subjected. These three effects still prevail.

Now what has all this to do with the psychology of learning? Everything. Most of it is wrong, because the facts of learning are being misstated through the influence of mechanistic assumptions. Mechanistic assumptions have been abandoned by twentieth century science all along the line. Assuming sense impressions to be the beginning of mental life, assuming learning to proceed by trial and error, assuming progress in any subject to follow laws of repetition, exercise, and effect, assuming mental processes to be composed of skills, is to impose false conceptions upon the facts, for these are mechanistic assumptions inherited from 1770 and 1860. The psychology of arithmetic that you learned is based upon these false assumptions and this, precisely, is why more pupils do not like arithmetic. It is not being taught correctly.

The change to a new educational psychology will not be made over-night, for the new, basic discoveries in the mother science, psychology proper, have yet to be applied wholesale in the classroom. Good teachers, however, have been applying it intuitively all along, as fast as they could, handicapped by a psychology that was mostly wrong. The same type of thing happened in all the sciences, for scientists as well as leaders in other fields are expressions of the age they live in and the training they receive. Individuals should not be blamed for the predicament that our teachers are in right now. The other sciences face the task of rewriting all their textbooks. And when they are rewritten the data, accumulated under mechanistic assumptions, will be redefined, reordered, reinterpreted. The painstaking work which these data represent will live again.

The main principles of the New Psychology. So much for history. Let us now review the main principles of the new psychology of learning.

1. Learning is a function of maturation and insight. It is a growth process that follows laws of dynamics, that is, laws of structured, unitary, energy systems or fields.[1]

2. First impressions are of total situations, but are undifferentiated. First movements are mass actions, likewise undifferentiated. In spite of appearances to the contrary, these impressions and movements are completely integrated.[2] (There is no such thing as lack of organization or of integration anywhere in the organic or the inorganic world.) But responses are sometimes unpredictable. Lack of predictability does not mean lack of coördination. Learning is a process during which unpredictability gives way to predictability, lack of control to control. The atom whirling around in a field of gas is behaving in a perfectly orderly fashion with respect to a constant flux in its surrounding field. The infant's hand, by wandering in its seemingly aimless fashion, is doing the same thing. The neuromuscular field is unstable. Once more, instability does not mean lack of organization. When the organization (perfect at all times) becomes static (the neuromuscular system has acquired stable form), then the hand movement becomes voluntary and predictable in ordinary situations. We erroneously say that coördination has been acquired. Only a different sort of coördination has been acquired, one that we now call "useful."

3. Learning is not exclusively an inductive process. First impressions are not chaotic and unorganized. They are merely unstable in the sense of not being under environmental control. There is

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[1] See R. H. Wheeler and F. T. Perkins, *Principles of Mental Development*, Thomas Y. Crowell Company, 1932. Several of these laws are explained in Chapter II.

[2] *Ibid.*, Chaps. III and IV.

nothing more highly organized than children's logic, to which impressions are subordinate. Adults do not discover this logic, that is all. Children flit from one thing to another, not because they are disorganized, but because their energy is in flux. That flux is as inevitably subject to law as a cyclone. And a cyclone obeys the same laws as does a steady breeze. So too, behavior in the child follows the same laws as the more stable behavior of the adult, only the conditions are different. There are laws of *wholes*, not laws of putting parts together.

Learning, then, is not a matter of forming bonds, a process of putting pieces of experience together.[3] It is not based on drill and on repetition of response. Bond psychology is irrational and has never been required by the facts of observation. It is a mechanistic, philosophy imposed upon the facts. On the contrary, learning is a logical process and from the beginning characterized by a grasp of relations, no matter how vague. Progress is systematic; it is a logical expansion and differentiation of unitary grasps of total situations—of wholes. It is organized and insightful, creative response to stimulus-patterns.

4. Learning does not proceed by trial and error. This concept is based upon an illusion, the fallacy of the double standard, arising out of the difference between the adult and the animal or child.[4] There is no such thing as a trial-and-error process anywhere in nature. Responses may be inadequate for two reasons: First, the general grasp of a purpose or a goal precedes adequate knowledge of how to reach the goal. The effort is orderly in spite of its failures. Failures occur because ideas are undifferentiated and therefore their outcome is not always predictable, even by the learner himself. Second, characterizing the effort of the learner as a failure is due to an illusion. A sophisticated person may be watching a beginner. The former knows the means to the end while the latter does not. He imposes his criteria upon the beginner, when those criteria do not apply.

5. More important, by far, than formal, prescribed methods of instruction are the personality of the learner and of the teacher, and the relationship between these personalities. Learning is subordinate to the growth and the demands of the personality-as-a-whole. The atmosphere of the classroom is more important than textbooks. The latter are necessary, but are secondary.

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[3] *Ibid.* See Chaps. XIII to XIX, inclusive, for experimental proof.

[4] *Ibid.*, pp. 356 ff.

6. Learning depends upon the will to learn, which cannot be forced by requirements or authority, but must be challenged by dynamic teachers and dynamic teaching.

7. Learning depends on clearness of goals, and the fitness with which tasks are adjusted to the pupil's level of maturation and insight. Progress is made by pacing.[5]

8. Goals are their own rewards, under natural law. Grades, grade points, many forms of motivation by social competition, and other hypocrisies are detrimental to learning. The subject must be worth learning in its own right. It can be made so, easily.[6]

9. A large part of the most efficient learning is incidental, that is, learning a special subject with reference to some broader interest or aim without realizing it: Learning number relationships in connection with telling time or making change; learning baseball averages (without effort) through sheer interest in big league contests; learning innumerable dates and incidents of history in the course of studying social and cultural trends. In none of these learning processes is drill involved.

10. Learning depends on transposition,[7] that is, discovery of form, system, order, pattern, logical relations, analogies, the repeated use of a hidden logical principle, and the making of relational judgments. It is not a matter of combining skills.

11. Subjects are inadequately learned when learned in isolation. Several should be learned together in terms of their interdependence.

12. No transfer will occur unless the material is learned in connection with the field to which transfer is desired. Isolated ideas and subjects do not integrate. Learning is not bond-forming. It is an orderly and organized process of differentiating general grasps of situations with respect to experience. The details emerge organized, as they differentiate from previous knowledge, in the face of new situations (not repeated ones).

Obviously detailed applications of the New Psychology to the teaching process await experimentation. Meanwhile, however, experimenters themselves, trained under a mechanistic definition of scientific method, must, of necessity, work under insurmountable handicaps until their conception of the experimental method, and what it is for, undergoes a radical change. Mechanistic nineteenth century science overestimated induction and underestimated deduction. It falsely rested upon a naïve empiricism that can be traced back to a British philosophy of the eighteenth century.

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[5] *Ibid.*, pp. 119, 197, 345, 381, 404, 415, 486.

[6] *Ibid.*, Chaps. XXII, XXIII, and XXVI.

[7] *Ibid.*, pp. 84, 454, 457, 458, 486.

The essence of the successful experiment is an adequate conception of the problem. Consider an illustration. A thousand children are taught beginning arithmetic by the configurational or logical method. Another thousand are taught by the association psychology, the drill method. It is found that the second group has better immediate retention of number combinations and that the first group understands number relations better and has better long-time retention. It is concluded that drill is better for immediate memory and should, therefore, be continued. This example is typical of the wasteful type of experimental work so characteristic of educational research. Such thinking simply does not face fundamental educational problems. It assumes that immediate memory is acquired by means of association, and remote memory through insight. This is nothing more nor less than a naïve imposition of two incompatible logics upon experiment.

Everyone knows that if nonsense syllables are to be memorized, they must be repeated. This is not because bonds are established by repetition, but because a column of nonsense syllables contains little more than space-and-time form, with a minimum of logical form, hence the total pattern is unstable. Numerous repetitions of the stimulus (not of the response) are necessary to effect and maintain stability of such a pattern. Time is wasted because the material is soon forgotten. Where learning material has little logical form, there is always short-time retention. For example, $2 \times 5 = 10$, $5 \times 5 = 25$, $5 \times 50 = 250$ have more logical form than $3 \times 9 = 27$, $6 \times 9 = 54$, $13 \times 17 = 221$. It is a sheer waste of time to drill pupils on combinations like the last three. They will not be remembered. When used they will be figured out.

The experiment we are discussing also makes the error of assuming that short-time retention is a goal of education, on the ground that it is of some value to the educative process and not to be supplanted by other values and methods. It is assumed that, when a method calculated to promote long-time retention is adequately presented, short-time retention does not follow as a matter of course. If retention is immediate only, it is certain that the material learned is meaningless to the pupil, and that drill will not give the material more meaning.

During mechanistic eras of science, the opinion prevails that scientific generalizations follow exclusively from inductive procedures, in short, that science is pure induction. This is only because interest centers on the parts which are supposed, naturally, to come first. Hence inductive experience is said to generate scientific laws. This is far from the case. No fundamental scientific principle has ever been discovered in this way, nor was this the claim of that great promoter of experimental science, Francis Bacon. Bacon was unalterably opposed to faith in theories that were not tested by observation, but he was not opposed to deduction.

Order of scientific method. Fruitful scientific experiments always have their beginnings in a well-thought-out idea, whose plausibility accrues to its apparent logical or deductive soundness. Neither experiment nor measurement, as such, leads to laws. A conception of a law or principle is necessary for the adequate planning of an experiment and for the choice of measuring instruments. Any educator who does not know this simple truth simply does not know the history of science. The order of the adequate scientific method, then, is this:

1. From a general understanding of nature, often suggested by a single experience, or at best a small number of experiences, a concept of a law or principle is developed.

2. This principle guides the planning of the experiment and the choice of quantitative units, even the construction of the apparatus. Gas laws had to be conceived before the thermometer, with which we measure temperature, could be constructed. Laws regarding electricity had to be known before the voltmeter, by means of which resistance is measured, could be made.

3. The experiment is performed in order to ascertain if the predictions, specified in the conceived law, hold true under rigid conditions.

4. The law receives refined quantitative expression in terms of the data obtained, but is not discovered by means of repeated observations. Scientific authorities have never disputed this point, but it seems to be commonly believed by educators that an experiment somehow has the divine power of revealing truth. Truths are not *revealed* by experiment. They are *tested* by experiment after insight has revealed them.

If assumptions employed in devising an experiment are logically unsound, the experiment will not necessarily reveal the error. On the contrary, the error will lead to a misinterpretation of the results. The association theory led to mis-statements of fact in psychology for two hundred years without detection and at least during fifty years of experimental work. In experiments on animals, twenty-five years ago, transposition was noted as an exception to the usual fact (now known to be extremely important), but so fully was association assumed that the "exceptional" facts were merely regarded as interesting curiosities. They compelled no revolution in theory. Even today the mechanistic pattern of thinking is so strong that Pavlov, for example, can see one thing and believe another, in direct contradiction to his own facts.

More than one hundred years of experimentation in chemistry failed to reveal the logical error of defining an acid in terms of a fixed substance. A discovery of the error requires but a few seconds of logical thought. In fact the error was suggested by chemists around 1820, but did not receive its rigorous experimental substantiation until the twentieth century.

The error of association psychology is not discoverable by means of experiments planned under mechanistic assumptions. The assumptions keep on distorting the results because the problems are inadequately conceived. Education keeps on paying the price. Two criteria are necessary before adequate facts can be guaranteed: (1) logical consistency of the assumptions underlying the conception of the problem to be tested, for it is in terms of these assumptions that the facts are to be evaluated; and (2) success in predicting the facts, or consistency of results. If thought is inadequate, the facts are bound to be inadequate. Associationistic assumptions are logically self-contradictory.[8] They cannot possibly, therefore, form the basis for any adequate experiment. If such an experiment proves successful, it is because the experimenter did what he inferred he was not doing.

Before teaching methods can be adequately improved on the basis of experiment, most of our educators must face the task, then, of altering their fundamental ideas about the learning process; they must face the task, first, of mastering the *theory* of the new psychology. Otherwise, they will have no adequate working hypothesis to test by experiment. One cannot experiment scientifically without an hypothesis.

Also, teachers can, for the present, obtain help from an intensive study of the *Gestalt* theory—more help than from associationistic workbooks. With a knowledge of this theory, teachers can change their own methods, and can arrange the presentation of their own material in an orderly, interrelated fashion, to suit the occasion. I know many teachers who are doing it.

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[8] *Ibid.*, Chap. XIX.

HINTS TO TEACHERS OF MATHEMATICS

Now the mathematics teacher who wants to be told just what to do and how to do it, at a particular hour of the day, will be disappointed in the following section. There is no exact formula for teaching any given subject or for correcting poor learning in any particular pupil. Each difficulty and each pupil is a problem in itself, to be solved by the use of general knowledge, that is, by the application of laws to the particular case.

The child-teacher relation is more important by far than the internal machinery of how the mind arrives at four by adding two and two. The adding process is a creative discovery, not a mechanical juggling of special skills.[9] We can best spend our time, therefore, trying to verify and to expand that intuitive insight every good teacher has, than to discuss the so-called mechanics of mathematical skills when such skills do not exist.

1. Make a large scrapbook of folklore about number; collect number games that reveal the logic of number relationships; have a rich repertoire of dramatic and interesting incidents in which number solves an important problem—saves a life, makes possible an adventure, results in human advancement. Study the lives of primitive peoples and note how number, size, matching, form, geometry, symmetry, played a living rôle in their daily existence. Refer frequently to dramatic achievements of man made possible by a mastery of mathematics: engineering, warfare, control of electricity, astronomy, medicine.

2. Relate number to form, pattern, invention, history, geography, nature study—to every conceivable form of vital human activity at the appropriate level of difficulty. Do not try to teach arithmetic; teach discovery, life, and nature through arithmetic.

3. Prepare interesting problems which can be solved by means of arithmetic. Teach arithmetic the way its discoverers learned it.

4. Forget drills. Prepare your work logically and concentrate on relations.

[9] *Ibid.*, Chaps. XXIV and XXV.

5. Teach numbers as operators. Dramatize arithmetic. Personify it. Explain in different ways what 2 will do to 4; what 5 will do to 10. Will any number whatever do anything to zero?

6. Use transposition. This is the systematic way to teach. Vary your problems or examples so that they will bring out an invariant number relationship, and select illustrations from a wide range. It is form—system—that is important. The arithmetic teacher who does not know the prevalence of form, rhythm, symmetry, repeating series, groups, etc., among number relationships should not attempt to teach arithmetic. She does not know her subject.

7. Study arithmetic and geometry, curves and graphs, for the life values and the logic they contain or imply. Learn to like mathematics yourself. It cannot be taught unless it is loved, unless it is one of the avenues for the expansion of the human soul. But to be loved, it must be understood. In your own study of mathematics, give it an *ethical* and *aesthetic* value, then breathe that atmosphere into the classroom. Number is intimately related to beauty and form.

8. The mathematics teacher will learn better how to teach mathematics by taking cultural courses and by deliberately observing the indirect part played by quantity in human affairs generally than in all the Education courses offered throughout the country.

9. Do not think that success as a mathematics teacher depends upon following a schedule sheet or an outline which tells you what to do at half-past ten on Tuesday morning. It does not. Nor is success measured by how many pupils you can prepare for the bimonthly standardized test. Do not prepare pupils for an examination. Instead, teach arithmetic. Some day the examinations will be abandoned!

10. Forget that you must eventually give pupils a grade. Teach arithmetic as something worth while in its own right.

11. In order to obtain a feeling for mathematics, read popular histories of mathematics, such as Bell's *Queen of the Sciences* and Dantzig's *Number, the Language of Science.* A proper feeling for the subject is a first step in becoming a successful teacher.

12. Ask unusual questions from time to time to find out how much the pupil really understands about a particular number or process. To illustrate: How near one is two? How long would it take to count to a hundred, a thousand, a million? How many minutes in a week? How far is it to the moon? How many people

are there in the world? Could they all get into the State of (the home state of the pupil)? Can a dog count? Would a mother dog know it if you took away one of her puppies? Why is a dozen instead of ten used as a common unit of sales? How would you spend a hundred dollars?

13. In the lower grades interest in mathematics can be helped by referring to such problems as catching busses, knowing when to come in for supper, when to go to bed, how often to feed the cat, how much to feed it. Plan discussion periods devoted to whens, and how-muches, in various situations of life.

14. Remember that knowing *what* to teach comes first. *How* to teach comes second.

15. If you wish truly to understand simple arithmetic and algebra study the great masters of higher mathematics—Cantor, Weierstrass, Dedekind, Peano, Hilbert, Bertrand Russell. Study them under a mathematics teacher who is philosophically minded, and can reduce the great discoveries of these masters to simple terms.

16. Eliminate from your mind that mathematics is, first, the science of number, quantity, and measurement. It is not. Primarily, it is a rigid logic, a science of precise order, pattern, transposition, invariants, matching. Study such phrases as the following, all of which are taken from the history of mathematics: the part is equal to the whole; the part has the power of the whole; you can prove the special case only when the general case is subject to proof; divide means contain; infinity is not a noun, it is an adjective; there are no infinitesimals; a "point" is in reality a "system"; a given number is a class. The logic of these assertions is the same as the logic of *Gestalt* psychology.

Teaching elementary arithmetic successfully. Success in elementary arithmetic depends upon the differentiation and expansion of an original, vague grasp of magnitude. Remember that primitive man perceived number-groups before he could comprehend additive aggregates. For example, 1, 2, 3, 4 meant unities of those magnitudes. Thus such words as *pair, duet, brace, trio, quartet, herd,* and *flock* are older than 2, 3, 10, 25. Moreover, 2 (as a single group) of one thing was different from 2 of another thing, hence a *couple* of persons, a *pair* of hands, a *brace* of partridges.

Modern children develop their concept of number in a similar way, but, of course, more rapidly. The first comprehension of number is relative, not absolute. It is one versus many, a few versus many, little versus large. If the number of units is not too large, magnitudes are compared and matched with surprising accuracy before the counting process starts.

Arithmetic should begin, therefore, with a study of matching and comparing, with no recourse to exact number at first and with problems so arranged that quantity can be differentiated from size and color. For example, the children might compare a group of three small blocks with a group of three large blocks, or a group of three red blocks with a group of three blue blocks, all the same size.

Ability to count is no guarantee whatever that the child knows where 2 or 8 is, in the range from 1 to 10. The process involves at first only an undifferentiated idea of succession, and, with respect to specific numbers, is absolutely mechanical. The individual numbers are not differentiated meaningfully until the child can count backwards and forward readily, from any starting-point in the range. All talk about kinaesthetic cues, auditory imagery, and the like, is utter nonsense, and quite irrelevant.

The first number to concentrate upon should be an easy one such as 4, 10, or 12. *Then the procedure in any case should be that of finding what the number contains.* This is ascertained by a simultaneous study of division, subtraction, multiplication, and addition. A longer time should be spent studying, say, 4. Use objects at first. The project is to find out what 4 means: $4 = 2 + ?$ $4 = 2 \times ?$ $4 \div ? = 2$ $4 = 1 + ?$ $4 = 3 + ?$ The general rule is, at first, to reverse the usual form of the equation, that is, the *whole should be presented first.* This method is slower at the outset but more effective in the end, and in fact, necessary, if the fundamentals of arithmetic are ever to be learned.

While exploring the range from 1 to 10, help to differentiate the range by using systematic games, bringing out the "secrets" of the series as a whole. For example, in addition and subtraction, study such series as the following:

$2 - ? = 1$
$3 - ? = 1$
$4 - ? = 1$
$5 - ? = 1$
$6 - ? = 1$

$10 - ? = 1$
$10 - ? = 2$
$10 - ? = 3$
$10 - ? = 4$
$10 - ? = 5$
$10 - ? = 6$

$1 + 1 = ?$
$1 + 2 = ?$
$1 + 3 = ?$
$1 + 4 = ?$
$1 + 5 = ?$
$1 + 6 = ?$

$1 + 9 = ?$
$2 + 8 = ?$
$3 + 7 = ?$
$4 + 6 = ?$
$5 + 5 = ?$
$6 + 4 = ?$

The teacher should keep referring to the range of numbers kept before the pupil on the board: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Many children learn mechanically to add and subtract without having any conception of the fact that 2 is next to 1 in the range, or that 8 is "2 numbers in from 10" or "7 positions over from 1."

The whole purpose of arithmetic is to discover number relationships and to be able to reason with number. It is not to learn the tables. The child should know how to derive one number from another in different ways. For example, how many ways can you obtain 5? $1 + 4$, $2 + 3$, 1×5, $5 - 0$, $6 - 1$, $10 \div 2$. Many children in the fourth grade are confused by this simple little question. They have not learned simple arithmetic, for it has not been taught to them.

It is quite as important to keep arithmetic alive by putting children into the problem situations, where, in order to get out, they must think in terms of number. After all, arithmetic should at first be taught as a special subject only at occasional intervals. It should generally be taught in connection with some interesting project.

Fractions, along with division should be begun early and scattered throughout the curriculum, as their difficulty permits. Introduce ½ when $4 \div 2 = 2$ is taught, ⅕ when $10 \div 2$ is taught. This is a good time to introduce $\frac{1}{2} + \frac{1}{2} = 1$. Teach $\frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$ of ½ by going back to $2 \times 2 = 4$.

MATHEMATICS IN THE JUNIOR HIGH SCHOOL

In the junior high school mathematics should be taught mainly as a means of making more precise a vitalizing and interesting knowledge of form and precision in nature. Study form and symmetry in the architecture of the savage and in his methods of calculating. How were the pyramids built? How was land surveyed in the days of ancient Egypt? Why were the Pythagoreans afraid of "irrational numbers," such as π? Study ratios and proportion as they were used in the simple Greek art. Note that the more beautiful buildings are cheaper to build and are more desirable. This is an actual discovery of engineering science. Beauty means economy. Note symmetry in animal and plant life—curves, spirals, spheres, cylinders, orderly transformations from one curve to another as in the veins of plants. Study birds' eggs, frost crystals, snow crystals, salt crystals, naturally formed jewels, spider webs, bee cells. Look for precision and mathematical rules of a simple character found almost anywhere in nature. Attack the problem of rhythm. Demonstrate complex tones with their overtones; consonance with its dependence upon simple harmonics. Study seasonal activity of all kinds, and cycles, by means of graphs. Attack simple problems in social science where averages and curves of distribution are employed. Make the study of interest alive, by historical anecdotes. In the middle ages charging interest was thought to be a sin. Why?

Teach the transposability of number. Do not permit the pupil to obtain the idea, for example, that five is only *a* number; it is *any* number of 5 units, and the units can be of any size one wants—elephants, planets, solar systems. A number is a *class* of numbers. This will help the pupil immeasurably when he comes to algebra. Teach also the relativity of number. Five is just 5, not 4.9 or 5.1, but it becomes *smaller and smaller in relation* to 10, 22, 105, 10,000, etc., and *larger and larger* in relation to 4, 3, 2, 1, ½, ⅛, ⅒, etc.

A few remarks might be made *apropos* of long division. Trouble comes from not building up to it logically, and from not adequately showing that no really *new* problem is involved. Play guessing games in addition, carefully planned to build up a better knowledge of the number range. The range from 1 to 100, for example, possesses "nodal points." Study the range as a whole for these points. Construct a long, narrow chart with all the numbers from 1 to 100 arranged in sequence. Point out the "nodal points" 50, 25, 75, the rhythmic sequence of 10's, and other "strategic" numbers which are the products of a number of different combinations, such as 48 (4 × 12; 6 × 8; 2 × 24; 3 × 16) and 72.

In long division trouble comes because a grasp of the *relative* size of the two numbers is not sufficiently differentiated. Hence the pupil is unable to estimate, for example, how many times 16 goes into 73.

Much help will be gained by noting, on the long number range, just mentioned, where 16 × 2, 16 × 3, 16 × 4, and 16 × 5 would come. Moreover, there are many interesting and useful short-cut methods of calculating that might be explained to the class. In short, the secret of long division is to have the number range from 1 to 100 differentiated as-a-whole.[10]

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[10] Wheeler, Raymond H. and F. T. Perkins, *Principles of Mental Development*. Thomas Y. Crowell Company, 1932. See also the discussion of pacing, in connection with this problem.

The teacher of high school mathematics has, at best, an acute problem to face. From the first grade on through the eighth, an increasing number of pupils acquire a distaste for mathematics, even a fear of it. Moreover, textbooks, again, get the cart before the horse. For example, plane geometry is naturally an *abstraction from* solid geometry. Geometry should begin, therefore, with those simpler problems of "solid" space and forms that do not require too much rigor of proof. The pupil should study different ways in which three-dimensional space can be *divided* or differentiated. He should receive training in visualizing intersecting planes and in visualizing lines (partitions) within solids and should become accustomed to finding forms ordered *within* forms. More than this, as Bertrand Russell complains, it is stupid to be teaching Euclidean geometry to pupils, anyway. They do not live in a Euclidean world. It is a hindrance to further study. We live in a Riemannian and relativistic world. It would be better to teach high school pupils the logic of relativity (they can readily grasp its essentials) than to grind them through the proofs of numerous dry theorems—save those few that are necessary for later work in practical surveying, engineering, and the like.

The logic of relativity promotes a social point of view, a much more accurate view of the relation of the individual to the group than the absolutistic inferences of ordinary high school mathematics. This statement may sound astonishing, but it should not be in the least. The atomistic thought-pattern that constructed Euclidean geometry, mechanistic science, and the mathematics of infinitesimals also constructed mercantilism, utilitarianism, monarchies, competition as the source of biological and social evolution, mechanistic biology, rugged individualism, and association psychology; justified war, and laid Christianity on the shelf! The relativistic pattern, prevailing in 1650, 1820, and 1935, works with the part-whole problem; leads to an organic view of nature, even of mathematics; it leads to coöperation, idealism, social democracy, intelligent pacifism, an emphasis upon harmony, and the balanced adjustment of parts within equilibrated wholes. This is the pattern upon which *Gestalt* psychology is based. If high school pupils are to go out into the world with any idea of what that world is like, they must be given some idea of the part-whole problem. Eventually, instruction in mathematics will help to solve the problem.

Our schools, today, face one of the most acute crises in history.

In the sixteenth and seventeenth centuries science revolutionized human culture. It is doing so again in the twentieth century. The eighth grader, today, understands without fear, either what the college professor did not know or was afraid to accept in 1600. This is equivalent to saying that the eighth grader of tomorrow will have to understand the gist of Einstein's theory of relativity and many other problems of the same logical category.[11] Unless this happens we shall not have a civilization, for the simple reason that there will be a new civilization based on the new science, or no civilization at all. The part the teacher of mathematics must play in this process is to forget skills, to discover the logic that mathematics contains, and to teach it, for that logic pertains to the part-whole problem.

In this chapter my purpose has been to stimulate further study of *Gestalt* psychology and to call attention to the universal shift in human thought, of which *Gestalt* psychology is only a special case. The high school teacher, like all of us, is dissatisfied. He is groping for something. He is participating in a cultural revolution. He is living in an exciting age. He wants to help in rebuilding the educational system that it may fit the age in which it finds itself, thereby serving the needs of humanity. He can do his part best, not by asking for course outlines and specific instructions to follow mechanically, but by deep, serious, thought pertaining to first principles, both of mathematics and of psychology.

[11] Read A. Korzybski's *Science and Sanity*. The Science Press, 1933.

MAKING LONG DIVISION AUTOMATIC

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A GOOD computer works as automatically as possible so far as his actual computation is concerned. If he is efficient, each step in the computation is reduced to the plane of habit and he proceeds from one step to the next with as small an expenditure of thinking or reasoning as possible. By making his computation automatic he works far more rapidly than he would if it were necessary to think out each step before taking it.

Examples of automatic computation. A good example of an automatic procedure in computation is found in multiplying the numbers shown at the right. In this case the multiplicand is quickly multiplied by each figure of the multiplier, and partial products are placed automatically. The actual multiplication by 0 is omitted but its effect is taken into consideration by moving the next partial product an extra place to the left. The addition of the partial products easily becomes a habit. The entire procedure is substantially standardized, at least so far as adults are concerned. If a hundred different adults did this example they would all proceed practically in the same way. The same thing is true if the numbers in the above example are added or subtracted. In the case of subtraction the adults fall into several groups, each group proceeding according to the particular method of subtraction that may have been taught, but within the group the procedure is standardized.

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Short division. With respect to the operation of division, however, the situation changes. If we have a problem in short division like the one at the right the method is practically uniform if the adults have really learned to do short division without the use of "crutches," that is, if they have learned to make it a purely mental affair, writing only the quotient figures. There is a group that will work such a problem by the long division process and another group that will follow the short division process, using

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some such crutch as writing the numbers representing the remainders. However, each group is fairly uniform in the procedure which it follows. The point that deserves emphasis is that in multiplication, addition, and subtraction, the work is so carefully standardized that it is done automatically regardless of the particular example that may be presented.

Long division. When we come to long division, however, the situation seems to be quite different and, though the general plan of working an example in long division may remain somewhat uniform with a given individual, he still varies his procedure in accordance with the particular example upon which he may be working. Whereas a hundred individuals will multiply two numbers together in exactly the same way, these same individuals on a given example in long division may so differ in the details of their thinking that no two will follow exactly the same method of work throughout. The reason for this wide difference in practice is due to the fact that long division constantly requires the estimation of figures and the exercise of judgment; such situations, however, do not arise in the more standardized procedures in addition, subtraction, and multiplication. The difficult part of long division, of course, is that connected with those steps by which one finds a tentative quotient figure, correcting it if necessary, and then arriving at the conclusion that the right figure has been found. It is this feature of long division that makes it by far the most difficult operation with whole numbers, not only for children but also for adults. It is likewise this feature that leads adults to follow such widely different practices in working a given example. After making an extensive study for the past fifteen years of the actual methods and devices used by hundreds of adults in long division, the author has found that even though adults may have originally been taught some scheme of long division that was fairly well organized, they have, through their own experience with this operation, come to adopt devices which vary from the procedure originally taught them. Unfortunately many adults never were given very systematic training when they learned long division. As children they were plunged into many of the difficulties of long division within a relatively short time and each was obliged to work out his own system. The detailed instruction concerning the various difficulties of long division and the careful grading of exercises such as are found in modern arithmetics did not exist in the texts that these adults studied.

Saving time in long division. The various devices or procedures which adults finally adopt in long division are usually the result of their desire to obtain each quotient figure in as short a time as possible. Any one who has had much experience with long division knows that he can get the correct quotient figures finally by a process of trial and error. He can guess at a quotient figure and then multiply to see if his guess is right. But all this is time-consuming. Careful investigation shows that such procedures as have been built up by experience are wholly for the purpose of avoiding the trial and error process and of arriving at the correct quotient figures as quickly as possible, with the minimum amount of multiplying or of testing to determine whether the quotient figures are right. Every computer always seeks the path of least resistance, saving a step here and there whenever he can. If a system of long division could be devised which would give the correct quotient figures *on first trial* in a very high percentage of all cases, and which would greatly reduce the need for correcting figures, such a system would make this operation practically as automatic as are the processes of addition, subtraction, and multiplication. It would also make possible the saving of an enormous amount of time in working examples in long division. It is the aim of this article to present such a system.

Variation in Procedures Used in Long Division

In order to arrive at a thorough understanding of our problem it will be necessary to study at some length the procedures actually used by different groups of adults in working long division examples. A group of typical examples is given below.

Example 1. In working the example at the right many adults divide the partial dividend *by the first figure of the divisor* in order to find the quotient figure. Thus, to find the first quotient figure they divide 9 by 2, which gives 4 as a trial quotient. Since 4 is found to be too large, they then try 3 which is right. To get the next quotient figure they divide 12 by 2, getting 6, which is too large. They try 5 which is also too large. They then try 4 which is correct. It should be noted that the correct quotient figure was obtained each time only after two or three trials.

$$\begin{array}{r} 34 \\ 26\overline{)900} \\ \underline{78} \\ 120 \\ \underline{104} \\ 16 \end{array}$$

If the above example had had a divisor of 21, 22, 28, or 29 instead of a divisor of 26, the procedure for obtaining the trial quotient figure by dividing by the *first figure of the divisor* would still be followed by a certain group of adults. A similar practice would be followed for divisors from 41 to 49, 71 to 79, etc. For the sake of convenience this procedure by which trial quotients are obtained by dividing by the first figure of the divisor will be referred to hereafter as Rule I.

Example 2. Instead of following the method outlined in the example above, where the divisor is 26, some adults find the trial quotient in this example by dividing each partial dividend by 3, which is *one more than the first figure* of the divisor. In other words, they think of 26 as being near 30. To find the first quotient figure they divide 9 by 3, getting 3, which is correct. To find the next figure of the quotient they divide 12 by 3, getting 4, which is also correct. By following this method it is seen that the correct quotient figure is found each time *on the first trial*. Those who work the above example, where the divisor is 26, by finding the quotient figures by dividing by one more than the first figure of the divisor, follow this same plan if the divisor is 27, 28, or 29, that is, they divide by 3 to find each quotient figure. The reason that they do this is because 27, 28, and 29 are nearer to 30 than they are to 20. In fact, this group of individuals also follows this plan for any other two-figure divisor from 26 to 99 if the second figure of the divisor is 6, 7, 8, or 9. There is another group of individuals that follows this plan in a limited way, applying it only when the second figure of the divisor is 8 or 9, as in the divisors 78 or 59. In this case all other divisors, which means those ending in 1 to 7, are handled by these individuals as in Example 1, that is, by Rule I. In other words, this last group of individuals divides by the first figure of the divisor except in those cases where the divisor ends in 8 or 9, in which case these individuals divide by one more than the first figure. The procedure by which trial quotients are obtained in the case of certain divisors by dividing by *one more than the first figure* of the divisor will be referred to hereafter as Rule II.

```
      34
26) 900
    78
    120
    104
     16
```

The use of Rule II in long division has been the subject of more or less discussion among teachers. There are some teachers who favor using it, though the particular divisors with which it is to be used are not always agreed upon, largely due to the fact that these teachers have limited knowledge concerning the efficiency of this rule. There are other teachers who do not use Rule II at all, preferring in all cases to divide by the first figure of the divisor, as illustrated in Example 1 on page 253. The relative merits of Rules I and II, particularly when applied to divisors ending in 6 to 9, will be discussed later in this article.

Example 3. Regardless of how the quotient figures are obtained in situations such as those illustrated in Examples 1 and 2, it is found that the procedure changes in examples like the following:

$$\begin{array}{r} 47 \\ 11\overline{)518} \\ \underline{44} \\ 78 \\ \underline{77} \\ 1 \end{array} \qquad \begin{array}{r} 63 \\ 12\overline{)759} \\ \underline{72} \\ 39 \\ \underline{36} \\ 3 \end{array} \qquad \begin{array}{r} 36 \\ 25\overline{)911} \\ \underline{75} \\ 161 \\ \underline{150} \\ 11 \end{array}$$

In these cases no attempt is made to divide by the first figure of the divisor to obtain the quotient figures. Instead one applies his knowledge of the multiples of 11, 12, and 25. Most adults and children know the tables of 11's and 12's, hence they make use of the memorized multiples of these numbers just as they use the multiples of 9 in an example in short division with a divisor of 9. The multiples of 25, up to 10 × 25, are also quite generally memorized through acquaintance with our money system.

Example 4. The example shown here represents another special situation in long division, this situation occurring in the step where the second quotient figure, 9, is obtained. After obtaining the first quotient figure which is found to be 2, the partial product, 124, is subtracted, leaving a remainder of 60. When the next figure, 7, is brought down there is a new partial dividend of 607. To obtain the next quotient figure the procedure varies. Some persons think "$60 \div 6 = 10$" but, since no quotient figure can be larger than 9, they change 10 to 9. They then proceed to check the correctness of 9 by multiplying 62 by 9, comparing the product with 607. The important consideration in this case is the method by which 9 was obtained, which was to divide 60 by 6, getting 10, and considering it as 9. With many persons this is not a natural procedure and it often causes confusion. Children are just as likely to divide 6 by 6, getting a quotient figure of 1, as they are to divide 60 by 6, getting a quotient of 10. Even if children do get a quotient of 10, they have to be carefully taught to consider it as 9.

$$\begin{array}{r} 298 \\ 62\overline{)18476} \\ \underline{124} \\ 607 \\ \underline{558} \\ 496 \\ \underline{496} \end{array}$$

There is another group of individuals who do not divide 60 by 6.

Instead, before bringing down the next figure, 7, they observe that the remainder, 60, is almost as large as the divisor 62, that is, they observe that 62 goes almost once into 60. Hence they conclude that when the next quotient figure is brought down, making a partial dividend of 607, the quotient will be 9.

There is still a third group of persons who make a mental comparison of the partial dividend, 607, with the divisor, 62, observing that 607 is a little less than 10 times 62, that is, that 607 is a little less than 620. Hence they conclude that 62 goes into 607 about 9 times and then try that figure. It must be understood, of course, that when this figure, 9, is obtained, regardless of which one of the above procedures is followed, it is considered to be a fairly safe guess. There is no guarantee that 9 is right until it has been tested in some way or until the product, 9 × 62, is obtained.

In connection with an example like $29\overline{)\,260}$, in which the divisor ends in 9, it should be observed that when one divides 26 by 2, getting a quotient of 13, he is applying Rule I. This quotient 13 is then changed to 9 and finally to 8. If this example were worked by Rule II one would divide 26 by 3, getting the correct quotient 8 on the first trial. In the case of all divisors ending in 6, 7, 8, and 9, Rule II always gives a *single figure* for the quotient, whereas Rule I in certain cases like the one just considered gives a *two-figure* quotient.

Example 5. Another special situation is illustrated in the example shown here. After getting the second quotient figure and subtracting the partial product, 252, we have a remainder of 3. Bringing down the next figure, we have a new partial dividend of 32. Regardless of the procedure by which the first two quotient figures were obtained, most persons obtain the third quotient figure, which is 0, by simple inspection, observing that 32 is less than 36 and hence 32 cannot contain 36. If the other quotient figures were found either by Rule I or by Rule II, these rules are usually abandoned when it comes to a situation where the partial dividend is less than the divisor. In other words, inspection operates in this case and immediately gives the correct quotient figure, which is 0.

```
      5709
36) 205524
    180
     255
     252
       324
       324
```

Inspection would be used in a similar manner in a case like $36\overline{)\,38}$ where it is easily seen that 38 contains 36 once. In other words, it is so evident by inspection that the quotient is 1 that no attempt is made to apply either Rule I or Rule II.

Example 6. A specialized situation in long division is illustrated by the example at the right where the divisor is 26. Instead of using Rule I or Rule II in this case the computer thinks of 26 as almost equal to 25, assuming that the quotient figure for a divisor of 26 will be very close to the quotient figure for a divisor of 25. Hence he thinks, "There are six 25's in 166. I will try 6." The figure 6 is then tested by finding the product of 26 by 6. Similarly, in getting the next quotient figure he thinks, "There are four 25's in 108. I will try 4." A similar practice would have been followed if the divisor were 24 instead of 26. It is evident, of course, that the person who follows this method of estimating the quotient figures for the divisors 24 and 26 is one who has memorized the multiples of 25. Otherwise such a procedure has little advantage. The device illustrated here is really a special trick for the particular divisors 24 and 26. If the divisor were 21 or 22, Rule I would probably be used.

$$\begin{array}{r} 64 \\ 26\overline{)1668} \\ \underline{156} \\ 108 \\ \underline{104} \\ 4 \end{array}$$

Testing quotient figures. In the six examples given above, which illustrate the wide variation in the procedures used in obtaining quotient figures, attention has largely been centered upon the methods by which one arrives at the trial quotient figure. Some reference has been made to the testing of these trial figures, this testing being assumed to be the actual multiplication of the divisor by the trial quotient to see if the product obtained is one that can be used. There are, however, other methods of testing quotient figures, none of which has general acceptance but all of which have more or less of a following. These other methods present almost as many varieties as do the procedures by which the trial quotients are found, which merely further illustrates the fact that long division is not a standardized operation like that of addition.

In order to enlarge our picture of this wide variation in procedures in long division, we will now discuss the various methods which are used for testing quotient figures.

Methods of Testing Trial Quotient Figures

When one obtains a trial quotient figure it can be tested to see whether it is correct by actually multiplying the divisor by the trial quotient and making the following observations:

Method A. If the product obtained is *greater* than the partial dividend, the quotient figure is too large. Hence a smaller figure must be tried.

Method B. If the product is smaller than the partial dividend, one finds the difference between the two and then compares the remainder with the divisor. If this remainder is *smaller* than the divisor the quotient figure is correct. If this remainder is *equal to* or *larger* than the divisor, the quotient figure is too small and a larger figure must be tried.

These methods of testing quotient figures should always be applied for each quotient figure obtained regardless of the particular system of long division that is being taught. The pupil should be required always to make these tests so that such testing will become a habit with him.

In addition to Methods A and B described above, there are supplementary methods by which one can determine *mentally* whether a quotient figure is right without getting the *complete product* of the divisor by the quotient figure, as is required when Methods A and B are used. When a complete product is obtained, *even if the multiplication is done mentally,* it is credited to Method A or Method B, rather than being considered as coming under one of the supplementary methods which are described below. These supplementary methods are as follows:

Method C. This method of testing is illustrated by the example at the right and is sometimes applied by those who use Rule I exclusively for estimating quotient figures. All the work in this testing is to be done mentally. The steps are as follows:

$$\begin{array}{r} 47 \\ 36\overline{)1694} \\ \underline{144} \\ 254 \\ \underline{252} \\ 2 \end{array}$$

(1) By using Rule I, the pupil gets 5 as the trial quotient. He then thinks, "5 × 30 = 150. 169 − 150 = 19 remainder. 5 × 6 = 30 which is larger than 19, so 5 is too large. Try 4."

(2) He tests the quotient 4 the same way, thinking, "4 × 30 = 120. 169 − 120 = 49 remainder. 4 × 6 = 24, which is less than 49, hence 4 is right."

(3) After step (2), the pupil multiplies 36 by 4, getting 144. It should be noted that no work in step (2) gave the product 144. Hence steps (2) and (3) were both used in getting the figure 4.

(4) Using Rule I, the pupil divides 25 by 3, getting a trial quotient of 8. He then thinks, "8 × 30 = 240. 254 − 240 = 14. 8 × 6 = 48, which is larger than 14, hence 8 is too large. Try 7."

(5) The quotient 7 is tested thus: 7 × 30 = 210. 254 − 210 = 44. 7 × 6 = 42, which is less than 44. Hence 7 is right.

(6) The pupil then multiplies 36 by 7, getting 252.

It is evident that this process is a very intricate one and that its successful use depends upon a high degree of mental concentration on the part of the pupil. This plan exercises the same type of mental skill that is required in short division, hence it really assumes that short division has been taught before long division, for without training of a similar kind in short division a pupil would make little progress with this method. Because of its difficulty Method C is not extensively used. In the above work it is important to notice that the testing in step (1) avoided the necessity of multiplying 36 by 5, but the testing in step (2) did not save the necessity of multiplying 36 by 4 in step (3). Likewise step (4) saved a multiplication by 8 but step (5) did not save the necessity of multiplying by 7 in step (6). In other words, 6 steps were needed to do this example, steps (1), (2), (4), and (5) representing concentrated mental work of a type that is very fatiguing to pupils. In fact, steps (1), (2), (4), and (5) are much more difficult than steps (3) and (6). If the example is worked without this mental testing of quotient figures, the steps are as follows:

(1) After getting the trial quotient of 5, the pupil actually multiplies 36 by 5 and finds the product too large; hence 5 is too large, according to Method A on page 257.

(2) He then multiples 36 by 4 and finds the product satisfactory by Method B on page 258. Hence 4 is right.

(3) After getting the trial quotient 8, he multiplies 36 by 8 and finds the product too large, hence 8 is too large.

(4) He multiplies 36 by 7 and finds the product satisfactory, hence 7 is right.

This solution requires only 4 steps which are carried out automatically, compared with 6 steps when Method C is used, 4 of these 6 steps being very difficult. In other words, the use of Method C represents extra work which necessarily slows up the process of long division. Those who advocate Method C usually require that every quotient figure be checked mentally in this way before the actual multiplication is made; the only exception to this statement is in the case of those quotient figures which are found by inspection in accordance with Rule III, described on page 263, or which are found by using one's knowledge of the multiples of 11, 12, and 25, as is explained under Rule IV on page 264.

Method D. Another method of testing quotient figures mentally

$$\begin{array}{r} 36 \\ 26\overline{)938} \\ \underline{78} \\ 158 \\ \underline{156} \\ 2 \end{array}$$

is shown here. After getting the first trial quotient, 4, the computer starts mentally to multiply 26 by 4, ignoring the unit's figure of the product and merely observing that the carry number 2, when added to 8 (4 × 2), gives 10, which is greater than 9 of the partial dividend 93. Hence 4 is too large. The computer then tries 3 which is first tested in the same way. This time the test shows that 3 is probably right, after which the actual multiplication, 3 × 26, is made to make certain of this fact.

This same type of mental testing is applied in getting the second quotient figure, which is first estimated as 7. By noting mentally that the carry number is 4 when 26 is multiplied by 7 the computer finds that 4 added to 14 gives 18, which is larger than the 15 of 158. Hence 7 is too large. He then tries 6 and tests it in the same way, after which the actual multiplication, 6 × 26, is made. In this work we see that the effect of the carry number is the important consideration, the actual unit's figure of the product being ignored. The unit's figure of the product is not determined in most cases until after this preliminary testing is finished. In other words, since the testing does not produce the complete final product, one has to go through an extra step to get it. If the mental testing had simultaneously given the complete product, this procedure would be classified under Method A on page 257 rather than under Method D. We see that mental testing by Method D requires extra steps and adds to the time needed in working examples in long division.

Method E. Another type of mental testing is illustrated by the example $64\overline{)489}$ where it is quickly seen that the quotient 8 is too large. This is really an easy application of Method C or Method D, depending upon how one does it. The reason for discussing this case separately is because some persons, who advocate using Rule I *for all divisors*, think that in a situation like this it is so self-evident that the quotient needs correction that they wish to consider this case as no more difficult than those in which the right quotient is obtained on first trial. In other words, they wish to count these particular "self-evident" cases as the equivalent of finding the right quotient at once, ignoring such mental testing as is done to determine that a correction is necessary. In this way they aim to show that the use of Rule I *for all divisors* is a more efficient procedure than this article will show it to be.

We do not consider it proper to count these "self-evident" cases as "rights," by ignoring the mental testing that is done. Skill in testing quotient figures mentally by Methods C, D, or E requires much time for children to acquire. Further, if a child limits this mental testing to particular types of cases like those above he must form the additional habit of examining each case to see if it is one of the kind to which he wishes to apply a mental test. All such work requires time and thought. Hence we believe there is no justification whatever for considering "self-evident" cases like those above as the equivalent of getting the right quotient on first trial.

From our point of view any kind of mental testing of a quotient figure, whether it be rapid or otherwise, must be considered as an *extra step* and so counted when one is determining the total number of steps needed to work a given example in long division.

The system of long division advocated later in this article assumes that methods of mental testing, like Methods C, D, or E, will not be used because they consume too much time. It is the elimination of such testing that makes it possible to have automatic long division. This question is discussed further on page 265 of this article.

Before discussing the next topic attention is called to the fact that in "self-evident" cases like those considered above there is a danger of assuming that the correct quotient is always 1 less than the trial quotient, it being understood that the trial quotient is obtained by the use of Rule I. It is readily seen that this assumption soon gives trouble and that these "self-evident" cases are not as simple as they seem. As evidence of this fact consider the following cases of this type where the true quotient is 2 or 3 less than the trial quotient: $26\overline{)186}$, $29\overline{)162}$, $38\overline{)243}$, $59\overline{)352}$. We grant in these cases that an experienced computer can see by a very rapid mental test that the first trial quotient is not the true quotient but it is not evident whether these quotients are wrong by 1, by 2, or by 3.

The need for automatic long division. It is very apparent from a study of the methods of obtaining and checking trial quotient figures given on the preceding pages that nothing like a standardized, automatic system for long division is followed by the great majority of people, corresponding to the standardized process that these same people employ in multiplying 497 by 341. It is also evident that each person is constantly endeavoring to find ways which will give him the correct quotient figure in the shortest time with the least amount of testing and multiplying. There are undoubtedly some methods in long division that are far more effective than others in obtaining quotient figures in certain situations. The difficulty seems to be that most adults have had no way of becoming acquainted with the many possible procedures in long division and of knowing their relative efficiency. They follow what they were originally taught, improving upon that method whenever possible on the basis of their own observation and experience. Most of them would gladly change certain of their practices in division if they could be convinced of the superiority of some other procedure over the one they are now using.

SYSTEM A—AN AUTOMATIC SYSTEM OF LONG DIVISION

In order to make it possible for one to proceed in long division as automatically as he does in the other fundamental operations with whole numbers, the author has devised a system of long division, which, for convenience, will be referred to hereafter as System A.

Efficiency of this system. This system of long division gives the following remarkable results in finding quotient figures:

(a) In 80.43% of all cases this system immediately gives the correct quotient figure on the first trial, that is, the first trial quotient is the true quotient.

(b) In 19.29% of all cases this system gives a quotient figure on the first trial that differs from the correct quotient figure *only by 1*. This means that only one correction has to be made in order to get the right quotient figure.

(c) In 0.28% of all cases this system gives a quotient figure that differs from the correct figure by 2. This means that in only 1 case out of every 357 cases is it necessary to make two corrections in order to get the right quotient figure.

(d) In no situations does this system produce a quotient figure that differs from the correct figure by more than 2. In other words, more than two corrections of the trial quotient figure are never necessary and two corrections are necessary only in an extremely small number of cases.

Summarizing the above results, it is seen that in 99.72% of all the cases this system gives *on first trial* either the correct quotient figure or a figure that differs from it only by one.

Details of the system. This highly efficient system as applied to two-figure divisors is made up of the following five rules:

Rule I. For divisors from 10 to 99 that end in 1, 2, 3, 4, or 5, each quotient figure is found by dividing the partial dividend by the *first figure* of the divisor.

Thus, for the divisor 43 the trial divisor is 4; for the divisor 85 the trial divisor is 8. The divisor 25 could be included under Rule I but, as explained later, it seems preferable to include it under Rule IV.

Rule II. For divisors from 19 to 99 that end in 6, 7, 8, or 9, each quotient figure is found by dividing the partial dividend by *one more than the first figure* of the divisor.

For example, if the divisor is 87 the trial divisor is 9; if the divisor is 49, the trial divisor is 5.

There are a few situations where Rules I and II are not applied, these situations being handled by Rules III and IV below.

Rule III. A rapid comparison of the divisor and the partial dividend should always be made to see if the quotient is *immediately evident.* If so, the quotient figure thus obtained should be used. This rule, which is called the method of *inspection* or *comparison*, may be applied with great rapidity in the following situations:

(a) In cases like $17\overline{)17}$, $23\overline{)24}$, $27\overline{)29}$, $34\overline{)39}$, $46\overline{)48}$, etc., where the partial dividend is *equal to* or *only a few units larger than* the divisor, it is immediately seen that the quotient is 1.

In the above examples it will be noted that the partial dividend is always in the same decade as the divisor. This assumption is made in order to make it easy for children to apply the method of inspection. Naturally for adults it will be easy also to apply this principle when the partial dividend is in a decade above the divisor. For simplicity, however, we make the limitation above noted. It is assumed that children will apply Rules I or II when the partial dividend is in a decade above that of the divisor.

(b) In cases like $17\overline{)13}$, $23\overline{)14}$, $29\overline{)17}$, etc., where the partial dividend is less than the divisor, it is *immediately seen* that the quotient is 0.

After long division has been fully learned, most persons intuitively apply Rule III first, before making other attempts to find the correct quotient figure. In teaching long division to pupils, however, it is easier to teach the use of Rules I and II before Rule III.

Some critics who have not given sufficient thought to the subject maintain that many pupils might apply Rule II to an example like

28)29, getting a quotient of 9 from 29 ÷ 3, instead of a quotient of 1. Hence they would abandon Rule II entirely, applying Rule I in all examples. In practice, however, no one applies Rule II in a case like 28)29. In such cases a rapid comparison of 28 and 29 *shows immediately* that the quotient is 1. The author has closely observed thousands of persons at actual work in long division and has yet to find a single person who would apply Rule II in a case like that above. All these persons, without exception, used Rule III in such examples. There are only a few other cases, such as 26)29, 37)39, 47)48, and 78)79, in which one could theoretically apply Rule II and obtain a quotient figure that differs considerably from the correct quotient. In all such cases it is seen that the partial dividend *never exceeds the divisor by more than 3 units.* Consequently, in all such cases, every one would intuitively use Rule III.

Rule IV. For the divisors 11, 12, and 25 it is assumed that the multiples of each of these divisors, up to 9 times the divisor, have been memorized and that these products will be used in obtaining quotient figures, just as one would use the multiples of 9 in short division with 9 as the divisor. Familiarity with the multiples of 25 comes through acquaintance with our money system while the tables of the 11's and 12's, or their equivalent, are usually learned in the third or fourth grade.

Rule V. The divisors 13 to 18 are treated separately just like six particularly difficult words in spelling. For these six divisors the quotients are found, in general, by the method of trial and error.

Thus, for each of these divisors we roughly estimate the number of times it will go into the partial dividend and then multiply to see whether this trial quotient is correct.

Some teachers apply Rule I to the divisors 13 to 18; that is, they divide the partial dividend by 1 to find the trial quotient figure. This practice is most tedious and unsatisfactory, since it requires several corrections of the trial quotient figure before the correct quotient is found. It should be noted that the divisor 19 is included under the divisors listed for Rule II.

Two important habits. In addition to the rules given above System A assumes that considerable attention will be given to having pupils establish the habit of comparing each new partial product with the partial dividend to be sure that no attempt is made to subtract a partial product which is larger than the partial dividend. This system also assumes that the pupil will establish the habit of comparing each new remainder with the divisor and to understand (1) that the quotient figure is correct if the remainder is less than the divisor, and (2) that the quotient figure must be made larger if the remainder is equal to or greater than the divisor.

Mental checking not required. In order that System A may be automatic it makes no use whatever of methods of testing quotient figures mentally such as those described under Methods C, D, and E on pages 258-261. In this system the pupil immediately multiplies his divisor by the trial quotient without stopping to make a mental test of the quotient figure. This multiplication may be done mentally or otherwise, but regardless of the way in which it is done it is regarded as a multiplication and counted as such when one is counting the total number of steps required to work a problem. Even though this multiplication may be a very easy one, it is still counted as a multiplication.

The reason that System A can discard methods of mental checking is because this system gives the right quotient figure on first trial in 4 out of every 5 cases. Why should we check every trial quotient figure mentally by Methods C or D when 4 out of every 5 quotient figures are right anyway? To discover that the quotient figure is wrong in the fifth case out of each five cases it is better to go ahead and make a multiplication that would have proved to be unnecessary if the quotient figures had been tested mentally rather than to waste time making 4 difficult mental checks in connection with 4 quotient figures that are right anyway. An illustrative problem in which the advantages of omitting mental checking are shown is given in connection with the discussion of Method C.

Three-figure divisors. Mention should be made of the fact that while System A as presented in this article refers to two-figure divisors, this system is equally efficient with divisors of three or more figures. In the case of larger divisors, the first two figures of the divisor indicate which rule is to be used. For example, in a divisor like 325 we use Rule I because the first two figures, 32, come under Rule I. Hence 3 is the trial divisor. Likewise, if the divisor were 383 we would use Rule II because 38 comes under Rule II. In this case 4 would be the trial divisor.

DETERMINING THE EFFICIENCY OF SYSTEMS A AND B

The method by which the efficiency of System A was determined is as follows. For each two-figure divisor all the partial dividends that can possibly exist for that divisor were studied in relation to Rules I, II, III, and IV. For example, the divisor 28 was studied with reference to its 280 possible partial dividends, these dividends running consecutively from 0 to 279 inclusive. It is thus seen that the largest partial dividend which can have 28 as a divisor is 279, it being understood, of course, that a partial dividend never contains the complete divisor more than 9 times. Hence the divisor 28 can not have partial dividends as large as 280 or 281. The largest partial dividend for any given divisor is 1 less than 10 times the divisor; the smallest partial dividend in each case is 0. Therefore the total number of partial dividends equals 10 times the divisor.

Method of studying the divisors. We will now give the details concerning the examination of each partial dividend for the divisor 28. For each partial dividend from 0 to 27 the quotient is 0, a fact which is immediately evident on inspection, as explained under Rule III on page 263. For the dividends 28 and 29 the quotient is 1, a fact which is also evident on inspection. This gives 30 cases in all from 0 to 29 where the correct quotient is found by inspection, that is, by Rule III. Hence the divisor 28 is credited with 30 "rights" which are entered in Column R in Table XII[1] on page 273.

We now continue the study of the divisor 28 in its relation to each partial dividend from 30 to 279, this study first being made with respect to Rule I. The results are recorded in Table VIII on page 269. Beginning with the dividend 30, we apply Rule I to this case, thinking 3 ÷ 2, which gives a quotient of 1, this quotient being correct. Hence in Table VIII, under Rule I, the divisor 28 is credited with 1 "right," this credit being for the dividend 30. In Table VIII the "rights" are placed in Column R. In the same way we examine the dividends 31 to 39 and obtain 9 more "rights," which are entered after the divisor 28 in Column R. When we study the dividend 40, however, the situation changes because 40 divided by 28 gives a quotient of 1, whereas Rule I applied to 40 gives 4 ÷ 2, or 2. Hence, in this case, Rule I gives a quotient that is wrong by 1, this fact being credited in Table VIII in the column

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[1] In Table XII the number of cases after each divisor includes all partial dividends from 0 up to the end of the decade in which the divisor appears; for example, for the divisor 52 it is assumed to be self-evident that the quotient is 0 or 1 for all partial dividends from 0 to 59 inclusive, 59 being the end of the decade in which 52 appears.

TABLE I

DIVISORS ENDING IN 1

Rule I			
Divisor	Total Cases	R	W_1
21	180	136	44
31	270	226	44
41	360	316	44
51	450	406	44
61	540	496	44
71	630	586	44
81	720	676	44
91	810	766	44
Totals	3960	3608	352
Per cent	100.00%	91.11%	8.89%

TABLE III

DIVISORS ENDING IN 3

Rule I				
Divisor	Total Cases	R	W_1	W_2
23	200	73	112	15
33	290	158	132	
43	380	248	132	
53	470	338	132	
63	560	428	132	
73	650	518	132	
83	740	608	132	
93	830	698	132	
Totals	4120	3069	1036	15
Per cent	100.00%	74.49%	25.15%	0.36%

R = Right on first trial. W_1 = Wrong by 1. W_2 = Wrong by 2.

TABLE II

DIVISORS ENDING IN 2

Rule I			
Divisor	Total Cases	R	W_1
22	190	102	88
32	280	192	88
42	370	282	88
52	460	372	88
62	550	462	88
72	640	552	88
82	730	642	88
92	820	732	88
Totals	4040	3336	704
Per cent	100.00%	82.57%	17.43%

TABLE IV

DIVISORS ENDING IN 4

Rule I				
Divisor	Total Cases	R	W_1	W_2
24	210	58	108	44
34	300	126	162	12
44	390	214	176	
54	480	304	176	
64	570	394	176	
74	660	484	176	
84	750	574	176	
94	840	664	176	
Totals	4200	2818	1326	56
Per cent	100.00%	67.10%	31.57%	1.33%

marked W_1, which means "wrong by 1." Similarly, in applying Rule I to each of the partial dividends from 41 to 49, we get 9 more quotients that are wrong by 1, these being entered also in Column W_1. We proceed in this way, examining each partial dividend in turn until we arrive at dividend 279. In some cases we find that Rule I gives a quotient that is wrong by 2, such cases being entered

TABLE V

DIVISORS ENDING IN 5

		Rule I			Rule II		
Divisor	Total Cases	R	W_1	W_2	R	W_1	W_2
25*	220	50	120	50	50	140	30
35	310	105	170	35	105	200	5
45	400	180	210	10	180	220	
55	490	270	220		270	220	
65	580	360	220		360	220	
75	670	450	220		450	220	
85	760	540	220		540	220	
95	850	630	220		630	220	
Totals	4060	2535	1480	45	2535	1520	5
Per cent	100.00%	62.44%	36.45%	1.11%	62.44%	37.44%	0.12%

* The divisor 25 is inserted for convenience but the figures following the divisor are not included in the totals at the bottom of the table since the divisor 25 is considered as coming under Rule IV.

TABLE VI

DIVISORS ENDING IN 6

		Rule I				Rule II		
Divisor	Total Cases	R	W_1	W_2	W_3	R	W_1	W_2
26	230	46	106	76	2	72	150	8
36	320	92	192	36		144	176	
46	410	156	244	10		234	176	
56	500	236	264			324	176	
66	590	326	264			414	176	
76	680	416	264			504	176	
86	770	506	264			594	176	
96	860	596	264			684	176	
Totals	4360	2374	1862	122	2	2970	1382	8
Per cent	100.00%	54.45%	42.71%	2.80%	0.04%	68.12%	31.70%	0.18%

in Column W_2. The dividend 128 is an example of this kind. If we actually divide 128 by 28 the true quotient is 4 but according to Rule I the quotient is 12 ÷ 2, or 6, which is wrong by 2. Hence the entry for 128 is put in the W_2 column. After completing the examination of all the dividends for the divisor 28 we find that we

TABLE VII

DIVISORS ENDING IN 7

		Rule I				Rule II	
Divisor	Total Cases	R	W_1	W_2	W_3	R	W_1
27	240	43	97	89	11	108	132
37	330	84	184	62		198	132
47	420	139	254	27		288	132
57	510	208	296	6		378	132
67	600	292	308			468	132
77	690	382	308			558	132
87	780	472	308			648	132
97	870	562	308			738	132
Totals	4440	2182	2063	184	11	3384	1056
Per cent	100.00%	49.15%	46.46%	4.14%	0.25%	76.22%	23.78%

TABLE VIII

DIVISORS ENDING IN 8

		Rule I				Rule II	
Divisor	Total Cases	R	W_1	W_2	W_3	R	W_1
28	250	42	88	96	24	162	88
38	340	78	172	90		252	88
48	430	126	256	48		342	88
58	520	188	312	20		432	88
68	610	262	344	4		522	88
78	700	348	352			612	88
88	790	438	352			702	88
98	880	528	352			792	88
Totals	4520	2010	2228	258	24	3816	704
Per cent	100.00%	44.47%	49.29%	5.71%	0.53%	84.42%	15.58%

have 42 cases in Column R, 88 cases in Column W_1, 96 in Column W_2, and 24 in Column W_3, all of this being shown in Table VIII. In all we have entered 250 cases in Table VIII, the remaining 30 cases being entered in Table XII. This accounts for the 280 partial dividends that belong to the divisor 28.

We now do the work for the divisor 28 all over again, examining the partial dividends 30 to 279 in relation to Rule II instead of Rule I. The results of this study are shown under Rule II in Table VIII. The partial dividends 0 to 29 continue to be handled by inspection and hence remain in Table XII.

TABLE IX

DIVISORS ENDING IN 9

		Rule I						Rule II	
Divisor	Total Cases	R	W_1	W_2	W_3	W_4	W_5	R	W_1
19	170	19	31	40	40	35	5	126	44
29	260	41	84	93	42			216	44
39	350	74	159	114	3			306	44
49	440	118	248	74				396	44
59	530	173	318	39				486	44
69	620	239	366	15				576	44
79	710	316	392	2				666	44
89	800	404	396					756	44
99	890	494	396					846	44
Totals	4770	1878	2390	377	85	35	5	4374	396
Per cent	100.00%	39.37%	50.11%	7.90%	1.78%	0.73%	0.11%	91.70%	8.30%

The above totals include the divisor 19.

We next study the divisor 38 in connection with all its partial dividends, this study being made first with reference to Rule III, then with reference to Rule I, and finally with reference to Rule II. We note that the divisor 38 has 380 partial dividends to be studied whereas the divisor 28 has only 280 such dividends. In fact, as the divisor becomes larger, the number of possible partial dividends increases. After the work for the divisor 38 is completed, the other divisors ending in 8 are studied in like manner. Table VIII gives the complete results for all the divisors ending in 8, except for the inspection cases which appear in Table XII. Examining Table VIII we find that Rule I gives results that are right in 2010 cases, wrong by 1 in 2228 cases, wrong by 2 in 258 cases, and wrong by 3 in 24 cases. It should be observed that the results are wrong by 1 more often than they are right. Comparing the 2010 "rights" with the total of 4520 cases studied in Table VIII, we find that Rule I gives the right quotient in 44.47% of the cases. Examining these 4520 cases with reference to Rule II we find from Table VIII that Rule II gives the right quotient in 3816 cases, which are 84.42% of all the cases. Comparing this 84.42% with the 44.47% obtained for Rule I we see that, for divisors ending in 8, Rule II is almost twice as efficient as Rule I with respect to the number of "rights" that it produces on the first trial.

Every other divisor from 19 to 99 was studied in the same way as the divisors ending in 8, the results being given in Tables I to IX on pages 267-270. Examining Table IX we find that for divisors ending in 9, Rule II produces "rights" on first trial in 91.70% of the cases, whereas Rule I produces "rights" on the first trial in only 39.37% of the cases. Thus we see that, for divisors ending in 9, Rule II is much more than twice as efficient as Rule I with respect to the number of "rights."

In Tables I to IV it should be noted that the divisors ending in 1 to 4 are studied only with respect to Rule I whereas in Tables V to IX the divisors ending in 5 to 9 are examined both for Rule I and for Rule II. The reason that we do not study the divisors ending in 1 to 4 with respect to Rule II is because no one would use that rule with this group of divisors.

Results of using Rules I and II. Using the figures given in Tables I to IX we are now able to compile Tables X and XI. Table X summarizes the results when Rule I is applied to all divisors ending in 1 to 5; Table XI gives the results when *both* Rule I and Rule II are applied to all divisors ending in 6 to 9. From Table X we see that Rule I has been applied to divisors ending in 1 to 5 in a total of 20,380 cases and that out of this number Rule I gives the right quotient on first trial in 15,366 cases, or in 75.40% of the total cases. From Table XI we find that Rule I has been applied to divisors ending in 6 to 9 in 18,090 cases and that out of this number Rule I gives the right quotient on first trial in 8444 cases or in 46.68% of the total cases. In other words, Rule I is far more effective when applied to divisors ending in 1 to 5 than it is when applied to divisors ending in 6 to 9, giving "rights" in 75.40% of the cases for divisors ending in 1 to 5 and "rights" in only 46.68% of the cases for divisors ending in 6 to 9. On the other hand, if Rule II is applied to divisors ending in 6 to 9, Table XI shows that it gives "rights" in 80.40% of the cases. It is very evident, therefore, that to use Rule I for divisors ending in 1 to 5 and to use Rule II for divisors ending in 6 to 9 gives much better results

TABLE X

SUMMARY FOR DIVISORS ENDING IN 1 TO 5

Divisors Ending in	Total Cases	Rule I		
		R	W_1	W_2
1	3,960	3,608	352	
2	4,040	3,336	704	
3	4,120	3,069	1036	15
4	4,200	2,818	1326	56
5	4,060	2,535	1480	45
Totals	20,380	15,366	4898	116
Per cent	100.00%	75.40%	24.03%	0.57%

TABLE XI

SUMMARY FOR DIVISORS ENDING IN 6 TO 9

Divisors Ending in	Total Cases	Rule I						Rule II		
		R	W_1	W_2	W_3	W_4	W_5	R	W_1	W_2
6	4,360	2374	1862	122	2			2,970	1382	8
7	4,440	2182	2063	184	11			3,384	1056	
8	4,520	2010	2228	258	24			3,816	704	
9	4,770*	1878	2390	377	85	35	5	4,374	396	
Totals	18,090	8444	8543	941	122	35	5	14,544	3538	8
Per cent	100.00%	46.68%	47.23%	5.20%	0.67%	0.19%	0.03%	80.40%	19.56%	0.04%

* The cases for the divisor 19 are included in this group both for Rule I and Rule II.

on the first trial than to use Rule I exclusively for all divisors. Such a combination of Rules I and II is used in System A.

Results of using Rule III. Having studied the results of using Rules I and II let us now study the results of using Rule III, by which quotient figures are obtained by inspection. The use of Rule III has already been described somewhat on page 263 and also on page 266 in connection with the study of the divisor 28. We will now consider this rule in its relation to all other divisors from 11 to 99 inclusive.[2] In connection with each of these divisors we examine those partial dividends which are such that it is clearly evident on inspection that they contain the divisor once, or that they do not contain it at all. In other words, for each divisor we consider those dividends where it is immediately evident that the quotient is either 0 or 1. For example, for the divisor 11, it is evident that the quotient is 0 for the partial dividends from 0 to 10. It is also evident that the quotient is 1 for the partial dividends from 11 to 19. We stop at 19 merely because 19 is in the same decade as the divisor 11, that is, we assume that it is easily seen that the quotient is 1 when the partial dividend is equal to or greater than the divisor, with the restriction that the partial dividends giving a quotient of 1 must be in the same decade as the divisor. With these assumptions in mind, let us apply them to several other divisors. For the divisor 21 the partial dividends which give a quotient of either 0 or 1 range from 0 to 29, making 30 partial dividends in all; for the divisor 85 these partial dividends range from 0 to 89, making 90 partial dividends in all; whereas, for 57 they range from 0 to 59, making 60 partial dividends in all. All this work, of course, comes under Rule III, which was described on

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[2] The multiples of 10 such as 20, 30, etc., are not included in this group since they are not regarded in this study as long division divisors.

TABLE XII

CASES WHERE QUOTIENT IS EVIDENT BY INSPECTION, USING RULE III

D = Divisor R = Number of cases that are right on first trial

D	R	D	R	D	R	D	R	D	R
11	20	fwd.	420	fwd.	1110	fwd.	2080	fwd.	3330
12	20	29	30	47	50	65	70	83	90
13	20	31	40	48	50	66	70	84	90
14	20	32	40	49	50	67	70	85	90
15	20	33	40	51	60	68	70	86	90
16	20	34	40	52	60	69	70	87	90
17	20	35	40	53	60	71	80	88	90
18	20	36	40	54	60	72	80	89	90
19	20	37	40	55	60	73	80	91	100
21	30	38	40	56	60	74	80	92	100
22	30	39	40	57	60	75	80	93	100
23	30	41	50	58	60	76	80	94	100
24	30	42	50	59	60	77	80	95	100
25	30	43	50	61	70	78	80	96	100
26	30	44	50	62	70	79	80	97	100
27	30	45	50	63	70	81	90	98	100
28	30	46	50	64	70	82	90	99	100
	420		1110		2080		3330	Total	4860

All cases given in the above table have quotients equal to 0 or 1.

page 263. Table XII on page 273 lists the number of partial dividends for each divisor where Rule III applies. From this table it is seen that there is a total of 4860 cases where it is immediately evident that the quotient is either 0 or 1. These 4860 cases are considered, therefore, as 4860 "rights." There are no cases under Rule III where the quotient is wrong by 1 or 2. Rule III, therefore, is a very satisfactory rule in that it always gives the right quotient immediately.

Results of using Rule IV. We will now consider Rule IV, described on page 264, which treats of the multiples of the divisors 11, 12, and 25. It is assumed that the pupil has memorized the multiples of 11 and 12 through the tables, at least up to 9 times each of these divisors. It is also assumed that the pupil knows the multiples of 25 through his acquaintance with our money system. If a child knows the multiples of 11, 12, and 25 he will apply them in long division in the same way that he applies the multiples of 7 in short division.

For the divisor 11 the total number of partial dividends that can exist is 110, these dividends ranging from 0 to 109. We have already assumed that for the divisor 11 the partial dividends from 0 to 19 will be handled by inspection, as is shown in Table XII on page 273. This leaves 90 partial dividends where it is expected that the memorized multiples of 11 will be applied. In these 90 cases, therefore, the right quotient will be obtained at once since we merely compare each partial dividend with the appropriate multiple of 11. Similarly, for the divisor 12 there is a total of 120 partial dividends, 20 of which have been assigned to Rule III, as shown in Table XII. This leaves 100 partial dividends where the multiples of 12 will be applied, these 100 cases giving the right quotient on the first trial. For the divisor 25 there are 250 partial dividends in all, 30 of which have been assigned to Rule III, leaving 220 partial dividends to be handled by the multiples of 25. Thus we have 220 more cases where the right quotient is obtained on the first trial. Summing this up, for the divisors 11, 12, and 25 we have 90 + 100 + 220, or 410 cases where memorized multiples will be used. Hence Rule IV will be credited with a total of 410 cases, all of which give the right quotient on the first trial.

Summary of System A. We have fully discussed Rules I, II, III, and IV and have shown how effectively these rules apply to the partial dividends that are assigned to them. We are now ready to combine the results with reference to these four rules in Table XIII which appears on this page. This table is really a summary of the system of long division advocated by the author, which we have designated as System A. Table XIII shows that System A comprises a total of 43,740 cases in long division and that this system gives the right quotient figure on first trial in 80.43% of all these cases, a quotient that is right within 1 in 19.29% of the cases, and a quotient that is right within 2 in 0.28% of the cases. In other words, this system gives the right quotient on first trial in 4 out of every 5 cases, the remaining cases having almost always a quotient that is right within 1. In making the statement that the remaining cases are almost always right within 1 we base it on the fact that quotients that are wrong by 2 occur in only 0.28% of all the cases which means that they occur in only 1 out of every 357 cases. To put this another way, in working problems in long division according to System A, only 1 out of every 5 quotient figures has to be corrected and, in general, only a single correction is needed. So far as the author knows, no other system of long division has ever been presented which is as effective and as practical as System A.

TABLE XIII

LONG DIVISION—SUMMARY OF SYSTEM A

	Total Cases	R	W_1	W_2
Rule I. (Divisors ending in 1 to 5.) See Table X.	20,380	15,366	4898	116
Rule II. (Divisors ending in 6 to 9 including 19.) See Table XI.	18,090	14,544	3538	8
Rule III. (Quotient evident on inspection.) See Table XII.	4,860	4,860		
Rule IV. (Multiples of 11, 12, and 25.) See page 264.	410	410		
Totals	43,740	35,180	8436	124
Per cent	100.00%	80.43%	19.29%	0.28%

Trial and error cases. The partial dividends connected with the divisors 13 to 18 have been included in System A only to the extent to which they come under Rule III, that is, only those partial dividends have been included for these divisors where the quotients are evident on inspection. The remaining cases relating to these divisors, which total 810 cases, are treated on the principle of trial and error since neither Rule I nor Rule II can be effectively applied to these cases without obtaining a large number of wrong quotient figures. This means that in the 810 cases just mentioned one guesses or estimates a quotient figure and then multiplies mentally or otherwise to see whether the figure is right. If the figure is wrong, it is corrected, being made either larger or smaller as the situation may require. If the remainder is larger than the divisor the pupil knows that the quotient figure must be made larger. If the product obtained is larger than the partial dividend the pupil knows that the quotient should be made smaller. It is important to note that in working problems by the trial and error method, the quotient figures sometimes have to be made larger and sometimes smaller, hence the pupil has to be acquainted with *two ways* of correcting quotient figures, this being true regardless of the system of long division that the pupil follows; for a further discussion of this matter, see page 283.

It is of interest to note that after one has gained some experience in working with the divisors 13 to 18, he memorizes certain of the multiples of these divisors. Most adults and many children will immediately tell you the product in such cases as 2 × 15, 3 × 15, 4 × 15, and 5 × 15. In fact, the multiples of the divisors 13 to 18 up to 5 times each of these divisors are usually remembered unconsciously after a certain amount of experience in computing. To the extent, therefore, that these products are memorized, the trial and error element disappears and the number of correct quotients obtained on first trial greatly increases. Hence, while theoretically there are 810 cases that we have assigned to be worked by trial and error, in practice it happens that about 400 of these cases are worked by using known multiples of the divisors, just as one works examples where the divisor is 11 or 12. This leaves only about 410 cases that have to be worked by trial and error. Even in these 410 cases the trial and error process can be speeded up considerably if children are taught mentally to multiply the divisors 13 to 18 by any number from 1 to 9, *beginning the multiplication with the ten's figure of the divisor* rather than with the unit's figure. For example, to multiply 17 by 5 mentally think "$5 \times 10 = 50$. $5 \times 7 = 35$. $50 + 35 = 85$." To multiply 18 by 7 think "$70 + 56 = 126$." After a little practice with this mental multiplication it is surprising to find how quickly it can be done.

Summary of System B. System A, which has just been described, applies Rule I to divisors ending in 1 to 5 and Rule II to divisors ending in 6 to 9. It is possible, of course, to perform the operation of long division without any use of Rule II whatever, making use only of Rule I, together with inspection and the multiples of 11, 12, and 25. Such a system of long division will, for convenience, be called System B to distinguish it from System A.

Let us now determine the efficiency of System B as a whole. In this connection we should first point out that System A consists of a total of 43,740 cases; this includes 170 cases[3] belonging to the divisor 19, which are handled by Rule II. In System B, however, it seems more sensible to include the divisor 19 in the group of divisors from 13 to 18 which are treated by trial and error. An examination of Table IX on page 270 will show that the divisor 19 gives very poor results if included under Rule I since it gives quotients that are wrong all the way from 1 to 5. Hence to include the divisor 19 under Rule I in System B would put that system at a distinct disadvantage. Further, no experienced computer would apply Rule I to the divisor 19. For these reasons System B will not include the divisor 19 under Rule I, hence it will have 170 cases less than System A. This will give System B a total of 43,740 − 170, or 43,570 cases. Those who follow System B will include the divisor 19 in the trial and error group.

TABLE XIV

LONG DIVISION—SUMMARY OF SYSTEM B

	Total Cases	R	W_1	W_2	W_3
Rule I. (Divisors ending in 1 to 5.) See Table X.	20,380	15,366	4,898	116	
Rule I. (Divisors ending in 6 to 9 omitting 19.) See Table XV.	17,920	8,425	8,512	901	82
Rule III. (Quotient evident on inspection.) See Table XII.	4,860	4,860			
Rule IV. (Multiples of 11, 12, and 25.) See page 264.	410	410			
Totals	43,570	29,061	13,410	1017	82
Per cent	100.00%	66.70%	30.78%	2.33%	0.19%

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[3] There are, of course, 190 partial dividends in all for the divisor 19, of which 20 are handled by Rule III. See Table XII.

The necessary facts concerning System B are given in Table XIV on page 277, from which it is evident that System B as a whole gives "rights" on first trial in 66.70% of the cases, quotients that are wrong by 1 in 30.78% of the cases, quotients that are wrong by 2 in 2.33% of the cases, and quotients that are wrong by 3 in 0.19% of the cases. The corresponding facts for System A as a whole are given in Table XIII on page 275.

COMPARISON OF SYSTEM A WITH SYSTEM B

A comparison of these two systems as a whole shows that System A gives "rights" in 80.43% of the cases while System B gives "rights" in only 66.70% of the cases. Likewise, System A gives quotients wrong by 1 in only 19.29% of the cases in comparison with 30.78% for System B. System A has cases wrong by 2 in only 0.28% of the cases in comparison with 2.33% for System B. System A has no cases wrong by 3 while System B has 0.19% of the cases wrong by 3.

It is shown above that the "rights" are 80.43% for System A and 66.70% for System B, making a difference of about 14% in favor of System A. This advantage of 14% becomes far more significant when we apply it to the total number of cases involved since it means that System A has over 6000 more "rights" than System B. Further, System A avoids much of the correction of quotient figures that is necessary in System B.

It is of interest to compare System B with System A not only in respect to the systems as a whole, as we have done above, but also with respect to certain aspects of these systems. Naturally, the only difference between these systems, with the exception of the treatment of the divisor 19, is that System A uses Rule II for divisors ending in 6 to 9 while System B continues to use Rule I for this group of divisors. So far as divisors ending in 1 to 5 are concerned both systems are identical. If the divisor 19 is left out of consideration, since it is treated differently in the two systems, we have 17,920 cases involving divisors ending in 6 to 9 which are treated by Rule I in System B and by Rule II in System A. Hence such superiority as System A may have over System B depends entirely upon the handling of these 17,920 cases. In any comparison, therefore, attention *must be centered particularly* upon this group of cases.

Comparing Rules I and II. A summary of the results of apply-

TABLE XV

COMPARISON OF RULES I AND II FOR DIVISORS ENDING IN 6 TO 9*

	Total Cases	R	W_1	W_2	W_3
Rule I	17,920	8425	8512	901	82
	100.00%	47.01%	47.50%	5.03%	0.46%
Rule II	17,920	14,418	3494	8	
	100.00%	80.46%	19.50%	0.04%	

* In this comparison the divisor 19 is omitted.

ing both Rules I and II to these 17,920 cases is given above in Table XV. Studying this table we find that out of 17,920 cases, Rule I gives 8425 "rights" on first trial, hence Rule I is right in 47.01% of the cases. When Rule II is applied to these 17,920 cases it gives 14,418 "rights," hence Rule II is right in 80.46% of the cases. It is apparent, therefore, merely by comparing the percentage of "rights" for each rule that the argument is decidedly in favor of Rule II for divisors ending in 6 to 9. This comparison, however, is further strengthened in favor of Rule II if we also study the cases that are not right. While Rule I is right in 8425 cases it gives a quotient that is wrong by 1, 2, or 3 in 9495 cases, which means that the "wrongs" are more frequent than the "rights." Attention is also called to the fact that in 983 cases the quotients are wrong by 2 or 3. In contrast, Rule II gives 14,418 "rights" and only 3502 "wrongs," practically all of these "wrongs" being wrong only by 1. The cases which are wrong by 2 are almost negligible since there are only 8 such cases out of a total of 17,920 cases, which means that in only 1 case out of each 2240 cases is the quotient wrong by 2. In view of these considerations there is not the slightest question with reference to the superiority of Rule II over Rule I when applied to divisors ending in 6 to 9. Further, it should be pointed out that there are still other considerations besides the statistical superiority just mentioned that give Rule II distinct advantages over Rule I, these considerations being discussed below.

Cases causing difficulty with Rule I. When Rule I is applied to divisors ending in 6 to 9 there are 2400[4] very awkward cases that arise where the application of Rule I to the partial dividend produces a two-figure quotient such as 10, 11, 12, 13, or 14 instead of a one-figure quotient, which we expect in long division. Examples of this situation are $28\overline{)205}$, which gives a quotient of 10, $26\overline{)227}$, which gives a quotient of 11, $39\overline{)364}$, which gives a quotient of 12, $28\overline{)268}$ which gives a quotient of 13, and $29\overline{)284}$ which gives a quotient of 14. In all these cases it is expected that the pupil will consider the quotient as 9 since a quotient figure larger than 9 is not possible.[5] Assuming that the quotient is 9 in such cases, the pupil first tries 9 to see whether it is right. If 9 is too large, he tries a smaller quotient. In many of these cases 9 is too large and has to be changed to 8, 7, or 6. The example $29\overline{)202}$ is an illustration of the case where the true quotient is 6 instead of 9. Since there are 2400 of these cases out of a total of 17,920 cases it is seen that about 1 out of every 8 cases involves this awkward situation where the quotient figure turns out to be a number anywhere from 10 to 14 when the partial dividend is divided by the first figure of the divisor. If Rule II is applied to this group of 2400 cases this embarrassment *never arises* since Rule II always gives a one-figure quotient and under no circumstances can it produce a two-figure quotient like 10, 11, or 14. Furthermore, when Rule II is applied to these 2400 cases it gives the correct quotient on first trial in 70% of the cases; when the quotient figure is wrong, it is practically always wrong by 1, there being only 8 cases where it is wrong by 2. This means that it is wrong by 2 in only 1 out of every 300 cases. Rule II never produces a situation where three corrections of the quotient figure must be made, as is sometimes necessary when Rule I is applied to these cases. These 2400 cases cause considerable difficulty not only to pupils but also to adults who are more experienced in the use of long division. With pupils the difficulty arises from the fact that they are not accustomed to get quotient figures like 10, 11, 12, or 14. In fact, in a case like $26\overline{)248}$ where the quotient figure must be considered as 12 it is more natural for the pupil to think $2 \div 2 = 1$, thus getting a quotient of 1 instead of 12. Due to the confusion that arises when Rule I is applied to these cases there is a tendency, especially with adults, to abandon

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[4] These 2400 cases do not include those belonging to the divisor 19.

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[5] Throughout this study in compiling the facts for Rule I in Tables I to XV a quotient of 10 or more has been considered as 9; if 9 was right, it was called a "right." On the other hand, if 9 was wrong, this result was counted "wrong by 1" or "wrong by 2" as the case might be.

Rule I when these situations arise and to handle these particular cases by inspection or by trial and error.

If we should deliberately take these cases out of System B and decide to handle them by trial and error, we would naturally put System B at a distinct disadvantage because we would greatly increase the trial and error cases. An efficient system of long division must give a definite result on the application of a given rule, hence such a system does not include any more trial and error cases than are absolutely necessary. The situation would not be so unfavorable with reference to these 2400 cases if Rule I applied to these cases produced quotient figures of only 10 and 11, but it does complicate matters when these quotient figures run up to 12, 13, and 14 because the child has first to think of 12, 13 or 14 as 9 and then often has to make other corrections. It should be emphasized again that these awkward situations can be avoided if Rule II is used since in all these 2400 cases Rule II gives only a single quotient figure.

It is true, of course, that there are also some situations of the kind just discussed that are connected with divisors ending in 1 to 5. Fortunately, however, there are only 1150 such cases[6] and in 1080 of these cases the quotient is always 10, the remaining 70 cases giving a quotient of 11. Another merit of these 1150 cases is that when we think of 10 or 11 as 9 it happens that 9 is the correct result in 1101 cases. In other words, we get 1101 "rights" out of 1150 cases. It is apparent, therefore, that the 1150 cases connected with the divisors ending in 1 to 5 that give quotients of 10 or 11 are far less numerous and much less complicated than the corresponding 2400 cases connected with the divisors ending in 6 to 9, which give quotients from 10 to 14 and which require many more corrections of the quotient figures.

We have already pointed out that these 2400 cases tend to encourage the users of Rule I to abandon that rule with respect to these particular cases and to use trial and error instead. The moment we encourage trial and error, we likewise encourage mental methods of checking quotient figures such as those described under Methods C, D, and E on pages 258-261. It has been made clear on those pages that mental methods of checking quotient figures are contrary to the interests of an automatic system of long division and greatly increase the amount of time necessary to do long division problems.

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[6] These 1150 cases do not include cases connected with the divisor 25.

Arguments against Rule II. In regard to Rule II the opponents of that rule argue, of course, that those who use Rule I never have to increase the first figure of the divisor by 1, this being necessary when Rule II is used; they seem to regard such an increase of the first figure of the divisor as something undesirable. While it does take a little time to form the habit of increasing the first figure of the divisor, this habit soon becomes a mechanical one and never requires an exercise of judgment or any skill in estimation like that which is necessary when one uses mental methods of checking quotient figures, such mental methods being frequently needed if one uses Rule I exclusively. It has always seemed amusing that the critics of Rule II are never disturbed in the slightest when they ask a child who uses Rule I to think of a trial quotient of 13 as 9 when it occurs in the many troublesome situations discussed on page 280. But these critics never cease to complain when one who advocates the use of Rule II asks a child to think of 59 as 60. In the former case the child really does not understand why he changes 13 to 9; he does it because he is told to do so. A child can readily understand, however, the reason for thinking of 59 as almost 60.

In deciding, therefore, whether to use Rule II we have to weigh carefully the consequences of not using it, the chief of which are the repeated corrections of the quotient figures that become necessary when only Rule I is used and the necessity of learning mental methods of checking quotient figures in order to reduce the number of corrections actually made. Certainly it is a relatively simple matter to form the habit of increasing the first figure of the divisor by one as compared with constantly facing the many annoyances that are bound to arise if Rule II is not used.

In considering any skill that is connected with long division we must remember that the value of that particular skill depends not only upon the frequency with which we use it in long division but it also depends upon how often it proves to be a skill that can be generally applied in various fields of arithmetic, that is, outside the field of long division. The habit of thinking of 29 as 30, or of 48 as 50, which is the predominant habit connected with the use of Rule II, is an extremely important one in arithmetic, being basically essential in all our work in "rounding off" numbers. When we are working problems involving money and get an answer like $1.579 we, of course, think of it as $1.580 or $1.58, which means that 79 is rounded off to 80. Similarly, when an advertisement in a newspaper states that hats are to be sold for $1.49 we know that this means practically $1.50. Likewise, if 29 children out of a class of 40 have defective teeth, we often think of $\frac{29}{40}$ as about $\frac{3}{4}$ since we think of 29 as approximately 30. The point to be emphasized is that frequently throughout arithmetic we are obliged to round off numbers and to get approximate answers. In fact, much stress is put upon this kind of thing in arithmetic today, particularly in connection with estimating in advance the answer to a problem, for all estimating involves the rounding off of numbers. It is apparent, therefore, that when we use Rule II in long division, which requires us to think of 57 as 60, or of 39 as 40, we are cultivating one of the most useful skills in arithmetic which in everyday life we will probably apply far more often in other situations than in long division.

The other argument that the opponents of Rule II so frequently present is that when Rule II gives the wrong quotient figure the correct quotient figure is always larger than the trial quotient figure, whereas when Rule I gives the wrong quotient figure the correct quotient figure is always smaller than the trial quotient figure. This means that if we use both Rule I and Rule II we have to become acquainted with two methods of correcting quotient figures, sometimes making them larger and sometimes making them smaller. It is urged that if Rule I is used exclusively only one method of correcting quotient figures is needed, which is to make them smaller, the assumption being that those who use Rule I exclusively never have any occasion to correct quotient figures by making them larger. This assumption is, of course, entirely without foundation because no one can build a system of long division in which all quotient figures are found by Rule I. Any system which uses Rule I exclusively also has a considerable need for the method of trial and error, which means that a quotient figure is tried and then made larger or smaller as the case requires. The facts are that there is far more trial and error used on the part of those who confine themselves to Rule I only than there is by those who use Rule I in conjunction with Rule II, that is, by those who use System A. The argument, therefore, that Rule I avoids the necessity of learning two ways of correcting quotient figures has little weight because one has to know these two ways when he works examples by the trial and error method.

It should be emphasized that when a pupil finds that a quotient figure is wrong he is never in any doubt whatever as to whether to make the figure larger or smaller if he has been given proper training with respect to two essential skills in long division, these skills relating to the comparison of each new partial product with the partial dividend and the comparison of the remainder with the divisor. These skills were described more fully under Methods A and B on pages 257 and 258.

There is still another consideration in connection with this matter of correcting quotient figures. Whether the figure be too large or too small, in general *only one correction* is necessary when System A is employed, whereas two or more corrections are frequently necessary when System B is used.

Summary of advantages of Rule II. Summing up this matter with respect to the advantages of Rule II over Rule I for divisors ending in 6 to 9, the following points deserve emphasis:

1. Referring to Table XV on page 279 we find that out of a total of 17,920 cases Rule II gives the right quotient figure on *first trial* in 80.46% of the cases whereas Rule I gives the right quotient figure in only 47.01% of the cases. Further, if the quotient figure is not right on first trial, Rule II requires, in general, only a single correction[7] of the quotient figure while Rule I may require two or three corrections.

2. Rule II avoids all the difficulties arising from 2400 cases where Rule I produces a two-figure quotient ranging from 10 to 14. In these cases Rule II gives a one-figure quotient. These 2400 cases represent 2400 out of 17,920 cases. Hence, if Rule II instead of Rule I is used for divisors ending in 6 to 9, difficulties are avoided that otherwise would arise in 1 out of every 8 cases.

3. Rule II gives pupils excellent training in "rounding off" numbers, which is a very useful accomplishment in arithmetic.

4. When Rule II is used with Rule I the pupil does not have to learn any more methods of correcting quotient figures than are necessary if he uses Rule I exclusively. Those who use only Rule I must know two ways of correcting quotient figures in order to work examples by the trial and error method, such cases occurring more often when Rule I is used exclusively than when both rules are used.

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[7] Rule II requires two corrections of the quotient figure in only 8 out of 17,920 cases, that is, in only 1 out of every 2240 cases.

ATTEMPTS TO IMPROVE SYSTEM B

All those persons who have had an extended experience with System B, which uses only Rule I, realize that they have a great deal of correcting to do in connection with their quotient figures. Hence we see various attempts to improve System B and at the same time to retain Rule I as far as possible.

Modifications of System B. One of the modifications of System B most frequently used is that in which Rule II is applied only to divisors ending in 9 while Rule I is applied to divisors ending in 1 to 8. Such a plan gives a system of long division in which there is a larger percentage of "rights" on the first trial than is possible when Rule I is used exclusively for all divisors. In this modified system the exact effect of applying Rule II only to divisors ending in 9 is shown by consulting Table IX on page 270. This table shows that there are 4770 cases in all to which divisors ending in 9 can be applied and that Rule II gives "rights" in 91.70% of these cases, while Rule I gives "rights" in only 39.37% of the cases.

Another modification of System B is to use Rule II for divisors ending in both 8 and 9 and Rule I for divisors ending in 1 to 7. The effect of such a modification is easily determined. We have shown above the advantages of using Rule II for divisors ending in 9. Hence we merely need to point out the additional gain that may come from applying this rule also to divisors ending in 8. From Table VIII on page 269 we find that the divisors ending in 8 include a total of 4520 cases and that Rule II gives "rights" in 84.42% of the cases while Rule I gives "rights" in only 44.47% of the cases. It is apparent, therefore, that a long division system gains in efficiency by applying Rule II rather than Rule I to divisors ending in 8 and 9 but it will not be as effective as System A, which applies Rule II to all divisors ending in 6 to 9.

From the above discussion it is easy to see why we frequently find people, and also textbooks, applying Rule II to divisors ending in 8 and 9. We might add that we usually find that those who follow the above practice are glad to extend the use of this rule to divisors ending in 6 and 7 when it is pointed out to them that, for divisors ending in 7, Rule II gives "rights" in 76.22% of the cases compared with 49.15% for Rule I (see Table VII); and that, for divisors ending in 6, Rule II gives "rights" in 68.12% of the cases compared with 54.45% for Rule I (see Table VI). It is largely ignorance of the facts concerning this matter that has kept many people from using Rule II for all divisors ending in 6 to 9, as is advocated in System A. We should note that for divisors ending in 5 the percentage of "rights" is about the same whether we use Rule I or Rule II, this fact being evident from an inspection of Table V on page 268. Hence it is purely a matter of personal preference whether Rule I or Rule II shall be used for the divisors ending in 5. System A uses Rule I for divisors ending in 5.

The author once knew a teacher who used Rule I exclusively but who had observed by experience that she did not have much success with this rule for divisors ending in 9. She knew that most of the quotients which she obtained were too large and hence needed correction so she decided that whenever the divisor ended in 9, she would reduce the trial quotient by 1 before multiplying the divisor by it. For example, if the trial quotient were 7, she called it 6. She had made no detailed study of the matter to determine whether this practice was justified but she did have a feeling that with it she got better results than otherwise. With the aid of Table IX on page 270 we can determine exactly how successful her plan would be. Under Rule I in that table we find for all divisors ending in 9, except the divisor 19,[8] that Rule I gives 1859 "rights," 2359 W_1 cases, 337 W_2 cases, and 45 W_3 cases. By her plan the 1859 "rights" would each become W_1 cases (with quotients too small by 1); the 2359 W_1 cases would become "rights"; the 337 W_2 cases would become W_1 cases; and the 45 W_3 cases would become W_2 cases. It should be noted that her plan gives two groups of W_1 cases. Adding these two groups we get 1859 + 337, or 2196 W_1 cases in all. It is evident, therefore, that this teacher did improve the efficiency of her work for divisors ending in 9 because she obtained 2359 "rights" by her method as compared with 1859 "rights" if she had followed Rule I in the usual way. She also obtained 2196 W_1 cases as compared with 2359 W_1 cases by the regular plan. She had the further advantage of changing 337 W_2 cases into W_1 cases, and 45 W_3 cases into W_2 cases. Summing it up she got more "rights" with less correcting to do than she would have had otherwise, though her gains were not outstanding in any way. The writer cannot recommend this method because it is quite inferior to the results obtained by the use of Rule II for divisors ending in 9. It does illustrate, however, the fact that those who use Rule I exclusively are not satisfied with it and are constantly attempting to do something to improve its efficiency.

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[8] The divisor 19 is omitted because it is assumed that one using Rule I exclusively would put the divisor 19 in the trial and error group.

HISTORICAL CONSIDERATIONS AND CONCLUSION

It will be of interest to consider Rule II historically for a few moments. Of course there is nothing new about this rule if we consider it merely as increasing the ten's figure of certain divisors by 1. In English and American textbooks in arithmetic published during the past hundred years we often find it recommended that for divisors like 58 or 59 we should use 6 as the trial divisor. Many of these textbooks do not give a specific rule stating the groups of divisors to which Rule II should be applied but they imply such a rule in the model examples which are worked out. It is in these models that the pupil is told, for example, to think of 59 as 60 and to use 6 as the trial divisor. There are other old books which clearly define just how Rule II should be used. For example, in De Morgan's arithmetic,[9] published in 1835, it is stated that when the second figure of the divisor is 5, or greater than 5, the first figure of the divisor should be increased by 1. In D. P. Colburn's arithmetic,[10] published in 1856, a similar rule is given which reads as follows:

> When the divisor is a large number, it is often convenient or necessary to use the nearest number of tens, hundreds, or thousands, as a trial divisor, to determine the probable quotient figure. For example, in dividing by 31, 32, 33, or 34 we make 30 or 3 the trial divisor. In dividing by 36, 37, 38, or 39 we make 40 or 4 the trial divisor. In dividing by 35 we may make either 30 or 40 the trial divisor.

It should be noted that Colburn leaves it to the choice of the computer as to what rule he shall follow for divisors ending in 5. Table V on page 268 of this article clearly indicates that there is practically no preference of one rule over the other for divisors ending in 5.

An interesting application of the practice of increasing the first figure of the divisor by 1 occurs in the arithmetic of Warren Colburn[11] published in 1822. This practice is illustrated in a case

[9] De Morgan, Augustus. *The Elements of Arithmetic.* 3rd edit. London, 1835.

[10] Colburn, Dana P. *The First Book of Arithmetic.* Philadelphia, 1856.

[11] Colburn, Warren. *Arithmetic; A Sequel to First Lessons in Arithmetic.* Boston, 1822.

like $36\overline{)197}$. Dividing by the first figure of the divisor Mr. Colburn thinks 19 ÷ 3 which gives 6. Then dividing immediately by one more than the first figure of the divisor he thinks 19 ÷ 4 which gives 4. The upper and lower "limits" of the trial quotient, as he expresses it, are 6 and 4, which suggests to him to try 5 as his first trial quotient.

We see, therefore, that the practice of increasing the first figure of the divisor by 1 in certain cases, as is represented by Rule II, has had approval for many years. The main difficulty in teaching this practice in the past has been a lack of definiteness as to whether it should be limited to divisors ending in 8 and 9 or whether it should include those ending in 6 and 7 as well. This lack of agreement was undoubtedly due to the fact that no one had taken the trouble to study this matter from all points of view, both statistically and otherwise, and to make his findings available, or that, if someone had made such a study, he did not publish his results. Further, regardless of the question as to whether Rule II should be applied to all divisors ending in 6 to 9 or to only a part of this group, there was no definite instruction showing how to avoid difficulties that might arise if Rules I and II were used in connection with certain cases to which neither of these rules should ever be applied. In other words, the use of inspection and of known multiples as devices to take the place of Rules I and II for certain cases, and a provision for the divisors 13 to 18 through the method of trial and error, had not been made, so far as the author knows, until he developed System A in one of his college classes about ten years ago. This development involved an enormous amount of painstaking work. Since then several others have studied certain aspects of this problem and have published their conclusions.[12]

It should be made clear that such contribution as System A may make to the teaching of long division does not lie wholly in the use of Rule II for the divisors ending in 6 to 9 but in a certain combination of Rules I and II with other devices. By clearly defining the areas in which the several procedures should be applied, System A avoids many awkward situations that otherwise might arise. By eliminating mental testing, System A also makes it possible to perform long division automatically. System A, therefore, is not only theoretically an efficient system but it is a thoroughly practical one as well. For the past eight years this system has been used very successfully in hundreds of elementary school classes.

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[12] See the following articles:

Knight, F. B. Comments on Long Division. *Fourth Yearbook*, Department of Superintendence of the N.E.A., p. 208. Washington, 1926.

Jeep, H. H. A Discussion of Long Division. *Second Yearbook, National Council of Teachers of Mathematics*, p. 41. New York, 1927.

Grossnickle, F. E. A series of three articles on long division. *Elementary School Journal*, Vol. XXXII, pp. 299, 442, and 595. December, 1931; February, April, 1932.